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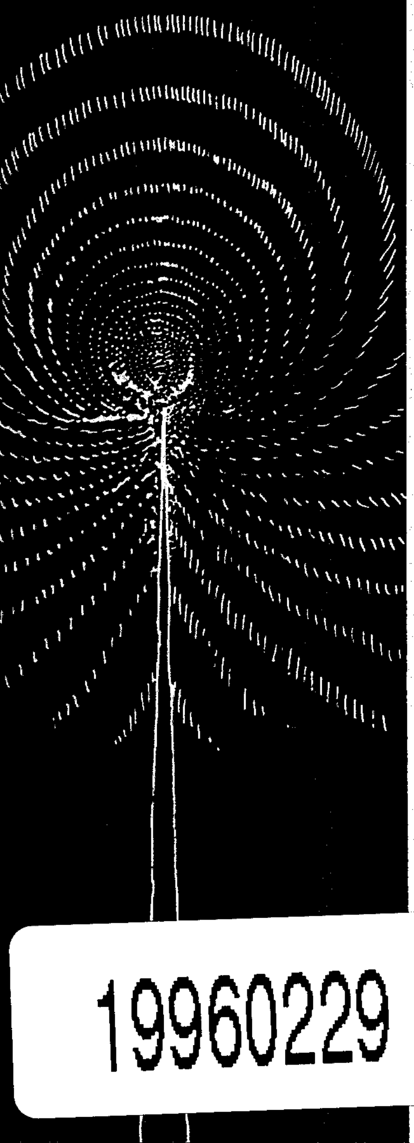

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NORMAL AND RADIAL IMPACT OF COMPOSITES WITH EMBEDDED PENNY-SHAPED CRACKS

BY

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16. Abstract

A method is developed for the dynamic stress analysis of a layered composite containing an embedded penny-shaped crack and subjected to normal and radial impact. The material properties of the layers are chosen such that the crack lies in a layer of matrix material while the surrounding material possesses the average elastic properties of a two-phase medium consisting of a large number of fibers embedded in the matrix. Quantitatively, the time-dependent stresses near the crack border can be described by the dynamic stress intensity factors. Their magnitude depends on time, on the material properties of the composite and on the relative size of the crack compared to the composite local geometry. Results obtained show that, for the same material properties and geometry of the composite, the dynamic stress intensity factors for an embedded (penny-shaped) crack reach their peak values within a shorter period of time and with a lower magnitude than the corresponding dynamic stress intensity factors for a through-crack.

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FOREWORD

This research work deals with the normal and radial impact of composites with embedded penny-shaped cracks which represents a portion of the program supported by the NASA-Lewis Research Center in Cleveland, Ohio. The program covers the period from February 13, 1978 to February 12, 1979 under Grant NSG 3179 and is conducted by the Institute of Fracture and Solid Mechanics at Lehigh University.

Professor George C. Sih served as the Principal Investigator while Dr. E. P. Chen was the Associate Investigator who is now employed by the Sandia Laboratory in New Mexico. The capable guidance of Dr. Christos C. Chamis who acted as the NASA Project Manager is very much appreciated. His encouragement has led to the success of this work.

TABLE OF CONTENTS

LIST OF FIGURES

LIST OF SYMBOLS

a	- radius of crack
$A(s,p), B(s,p)$	- unknowns in dual integral equations
$A^{(i)}, B^{(i)}, C^{(i)}$	- coefficients for transform of solution, functions of (s,p)
b	- half of the thickness of the layer
Br	- Bromwich contour in the complex p-plane
c_{1j}, c_{2j}	- dilatational and shear wave speeds for medium j
$e^{(i)}$	- functions of (p,s) through γ_{ij}
$f^*(p)$	- Laplace transform of $f(t)$
$f^h(s)$	- Hankel transform of $f(x)$
$(f)_j$	- indicates that f is evaluated in medium j
$h(t)$	- Heaviside unit step function
$J_n(x)$	- Bessel function of order n
$k_1(t), k_2(t)$	- dynamic stress intensity factors
$M_I(\xi,\eta,p)$, $M_{II}(\xi,\eta,p)$	- kernel of Fredholm integral equation
$P_I(s,p), P_{II}(s,p)$	- kernel in dual integral equations
r, θ, z	- cylindrical coordinates
r_1, θ_1	- crack tip polar coordinates
u_r, u_θ	- displacement components
t	- time
x, y, z	- rectangular coordinates - crack lies in the xy-plane
γ_{ij}	- exponents for transform of solution, functions of (p,s)
$\delta^{(i)}$	- functions of (p,s) through $e^{(i)}$
Δ_I, Δ_{II}	- functions of (p,s) through $\delta^{(i)}$
λ_1, λ_2	- Lamé coefficient

$\Lambda_I^*(\xi,p),\Lambda_{II}^*(\xi,p)$	- unknown in Fredholm integral equation
μ_1,μ_2	- shear modulus
ν_1,ν_2	- Poisson's ratio
ρ_1,ρ_2	- mass density
σ_o	- suddenly applied normal stress
$\sigma_r,\sigma_\theta,\sigma_z,\tau_{rz}$	- stress components
τ_o	- suddenly applied shear stress
ϕ_j,ψ_j	- scalar potentials for medium j
∇^2	- Laplacian operator

NORMAL AND RADIAL IMPACT OF COMPOSITES

WITH EMBEDDED PENNY-SHAPED CRACKS

by

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ABSTRACT

A method is developed for the dynamic stress analysis of a layered composite containing an embedded penny-shaped crack and subjected to normal and radial impact. The material properties of the layers are chosen such that the crack lies in a layer of matrix material while the surrounding material possesses the average elastic properties of a two-phase medium consisting of a large number of fibers embedded in the matrix. Quantitatively, the time-dependent stresses near the crack border can be described by the dynamic stress intensity factors. Their magnitude depends on time, on the material properties of the composite and on the relative size of the crack compared to the composite local geometry. Results obtained show that, for the same material properties and geometry of the composite, the dynamic stress intensity factors for an embedded (penny-shaped) crack reach their peak values within a shorter period of time and with a lower magnitude than the corresponding dynamic stress factors for a through-crack.

*This work was completed when Dr. Chen was a faculty member at Lehigh University.

INTRODUCTION

Advanced composite materials are multi-phased nonhomogeneous materials with anisotropic properties. This complicates the stress analysis for fracture, particularly if the loading is time-dependent and the geometry involves sharp edges such as a crack. As a result, conventional and mathematical techniques for dynamic fracture generally fail to yield accurate results.

An effective approach for finding dynamic stresses in a nonhomogeneous composite containing a through crack has been developed [1] by utilizing both the Laplace and Fourier transforms. The transient boundary, symmetry and continuity conditions were formulated by integral representations in terms of the rectangular Cartesian coordinates x and y and the results for the stress intensity factors are determined numerically by solving a standard integral equation in the Laplace transform plane. The crack geometry was assumed to be extended infinitely in the z-direction or through the side wall of the composite specimen. Many of the failures in composites, however, were observed [2] to initiate from embedded mechanical imperfections such as air bubbles, voids or cavities. Hence, a more realistic modeling of the actual flaw geometry would be an embedded crack that has finite dimensions in all directions. This immediately suggests a three-dimensional elastodynamic crack problem which cannot be solved effectively by analytical means unless symmetry prevails. One approach for obtaining a solution is to extend the integral transform formulation for a through crack in rectangular coordinates [1] to that of an embedded crack in cylindrical polar coordinates. This necessitates the use of Hankel transforms instead of Fourier transforms.

Although no attempt will be made to analyze the failure of the composite due to impact, the dynamic stress intensity factors $k_1(t)$ and $k_2(t)$ can be readily

used in a given fracture criterion, say the strain energy density theory [3], for determining the allowable level of impact load. The new results can also assist the construction of composite materials for establishing impact tolerance. In this case, failure is assumed to initiate from a damage zone of material in the composite that can be approximated by an embedded crack. The time-dependent characteristics of the stresses for the through and embedded crack geometries are compared and studied for different elastic properties and dimensions of the composite. In particular, the phenomenon of elastic waves reflecting from the crack to the interfaces within the composite can be exhibited numerically when their neighboring boundaries are sufficiently close to one another. As time becomes very large, all of the results in this report reduce to the corresponding static solutions [4].

AXIAL SYMMETRIC DEFORMATION: PENNY-SHAPED CRACK

Consider a penny-shaped crack of radius a that lies in a layer of material of thickness 2b with material properties μ_1, ν_1, ρ_1. This layer is bonded between two media with properties μ_2, ν_2, ρ_2 as illustrated in Figure 1. With reference to the system of coordinates (x,y,z), the z-axis coincides with the center of the crack and is normal to the crack situated in the xy-plane. The outer boundaries of the composite are assumed to be sufficiently far away from the crack such that the reflected waves will have a negligible influence on the local stresses. Only those impact loads that produce an axisymmetric wave pattern will be considered.

For an axially symmetric deformation field, material elements are displaced only in the radial and axial direction and remain unchanged in the θ-direction. With reference to the cylindrical polar coordinates (r,θ,z) in Figure 1, the

two nonzero displacement components can be expressed in terms of the wave potentials $\phi_j(r,z,t)$ and $\psi_j(r,z,t)$ as follows:

$$(u_r)_j = \frac{\partial\phi_j}{\partial r} - \frac{\partial\psi_j}{\partial z}$$

$$(u_z)_j = \frac{\partial\phi_j}{\partial z} + \frac{\partial\psi_j}{\partial r} - \frac{\psi_j}{r} \tag{1}$$

where j = 1 refers to the layer with the crack and j = 2 to the surrounding material. The four nontrivial stress components are given by

$$(\sigma_r)_j = 2\mu_j \frac{\partial}{\partial r} (\frac{\partial\phi_j}{\partial r} - \frac{\partial\psi_j}{\partial z}) + \lambda_j\nabla^2\phi_j$$

$$(\sigma_\theta)_j = 2\mu_j \frac{1}{r} (\frac{\partial\phi_j}{\partial r} - \frac{\partial\psi_j}{\partial z}) + \lambda_j\nabla^2\phi_j$$

$$(\sigma_z)_j = 2\mu_j \frac{\partial}{\partial z} (\frac{\partial\phi_j}{\partial z} + \frac{\partial\psi_j}{\partial r} + \frac{\psi_j}{r}) + \lambda_j\nabla^2\phi_j \tag{2}$$

$$(\tau_{rz})_j = \mu_j [\frac{\partial}{\partial z} (2 \frac{\partial\phi_j}{\partial r} - \frac{\partial\psi_j}{\partial z}) + \frac{\partial}{\partial r} (\frac{\partial\phi_j}{\partial r} + \frac{\psi_j}{r})]$$

in which λ_j and μ_j are the Lamé constants and ∇^2 represents the operator

$$\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{\partial^2}{\partial z^2}$$

The governing equations can thus be obtained from the equations of motion which yield

$$\frac{\partial^2\phi_j}{\partial r^2} + \frac{1}{r}\frac{\partial\phi_j}{\partial r} + \frac{\partial^2\phi_j}{\partial z^2} = \frac{1}{c_{1j}^2}\frac{\partial^2\phi_j}{\partial t^2}$$

$$\frac{\partial^2\psi_j}{\partial r^2} + \frac{1}{r}\frac{\partial\psi_j}{\partial r} - \frac{\psi_j}{r^2} + \frac{\partial^2\psi_j}{\partial z^2} = \frac{1}{c_{2j}^2}\frac{\partial^2\psi_j}{\partial t^2} \tag{3}$$

with c_{1j} and c_{2j} being the dilatational and shear wave speeds:

$$c_{1j} = \left(\frac{\lambda_j+2\mu_j}{\rho_j}\right)^{1/2}, \quad c_{2j} = \left(\frac{\mu_j}{\rho_j}\right)^{1/2} \tag{4}$$

If the composite body is initially at rest, the Laplace transform of equations (3) further give

$$\frac{\partial^2\phi_j^*}{\partial r^2} + \frac{1}{r}\frac{\partial\phi_j^*}{\partial r} + \frac{\partial^2\phi_j^*}{\partial z^2} = \frac{p^2}{c_{1j}^2}\phi_j^*$$

$$\frac{\partial^2\psi_j^*}{\partial r^2} + \frac{1}{r}\frac{\partial\psi_j^*}{\partial r} - \frac{\psi_j^*}{r^2} + \frac{\partial^2\psi_j^*}{\partial z^2} = \frac{p^2}{c_{2j}}\psi_j^* \tag{5}$$

Here, p is the transform variable in the Laplace transform pair:

$$f^*(p) = \int_0^\infty f(t)\exp(-pt)dt$$

$$f(t) = \frac{1}{2\pi i}\int_{Br} f^*(p)\exp(pt)dp \tag{6}$$

The abbreviation Br stands for the Bromwich path of integration. Moreover, since the composite geometry is symmetrical about the xy-plane, it suffices to consider

only the solution in the upper half-space, $z \geq 0$. For the penny-shape crack geometry, the Hankel transform pair [5] may be used:

$$f^h(s) = \int_0^\infty xf(x)\, J_n(sx)dx$$
$$f(x) = \int_0^\infty sf^h(s)\, J_n(sx)ds \tag{7}$$

where J_n is the nth order Bessel function of the first kind. Applying equations (7) to (5), the following results are obtained:

$$\phi_1^*(r,z,p) = \int_0^\infty [A^{(1)}(s,p)e^{-\gamma_{11}z} + A^{(2)}(s,p)e^{\gamma_{11}z}]\, J_0(rs)ds$$
$$\psi_1^*(r,z,p) = \int_0^\infty [B^{(1)}(s,p)e^{-\gamma_{21}z} + B^{(2)}(s,p)e^{\gamma_{21}z}]\, J_1(rs)ds \tag{8}$$

for the cracked layer and

$$\phi_2^*(r,z,p) = \int_0^\infty C^{(1)}(s,p)e^{-\gamma_{12}z}\, J_0(rs)ds$$
$$\psi_2^*(r,z,p) = \int_0^\infty C^{(2)}(s,p)e^{-\gamma_{22}z}\, J_1(rs)ds \tag{9}$$

for the surrounding material. The quantities γ_{ij} are given by

$$\gamma_{1j} = (s^2 + \frac{p^2}{c_{1j}^2})^{1/2}, \quad \gamma_{2j} = (s^2 + \frac{p^2}{c_{2j}^2})^{1/2} \tag{10}$$

The six unknowns $A^{(1)}$, $A^{(2)}, \ldots, C^{(2)}$ are determined from a given set of transient boundary, symmetry and continuity conditions.

NORMAL IMPACT

Let the penny-shaped crack be subjected to a uniform impact load* such that the upper and lower surface will move in the opposite direction. The magnitude of this normal load is σ_0 and since it is applied suddenly from t = 0 and maintained at a constant value thereafter, the Heaviside unit step function, H(t), will be used, i.e., $-\sigma_0 H(t)$. Making use of equations (6), the conditions on the plane z = 0 for $r \leq a$ and $r \geq a$ take the forms

$$(\sigma_z^*)_1(r,o,p) = -\frac{\sigma_o}{p};\ (\tau_{rz}^*)_1(r,o,p) = 0,\ 0 \leq r < a$$

$$(u_z^*)_1(r,o,p) = 0;\ (\tau_{rz}^*)_1(r,o,p) = 0,\ r \geq a \tag{11}$$

If the interfaces at z = ±b is bonded perfectly, the stresses and displacements can then be considered continuous across these planes, i.e.,

$$(\sigma_z^*)_1(r,b,p) = (\sigma_z^*)_2(r,b,p)$$

$$(\tau_{rz}^*)_1(r,b,p) = (\tau_{rz}^*)_2(r,b,p) \tag{12}$$

*There is no loss in generality in formulating the problem in terms of a uniform step load. The principle of superposition may be used to obtain the solution for general loading from a series of step loading solutions as discussed in [1].

and

$$(u_r^*)_1(r,b,p) = (u_r^*)_2(r,b,p)$$

$$(u_z^*)_1(r,b,p) = (u_z^*)_2(r,b,p) \tag{13}$$

Under these considerations, the six functions $A^{(1)}$, $A^{(2)}, \ldots,$ $C^{(2)}$ may be expressed in terms of a single unknown $A(s,p)$ as indicated by equations (A.1) in the Appendix.

Fredholm integral equations. Without going into details, the function $A(s,p)$ can be obtained from the system of dual integral equations

$$\int_0^\infty A(s,p)\, J_0(rs)ds = 0, \quad r \geq a$$

$$\int_0^\infty sP_I(s,p)\, A(s,p)\, J_0(rs)ds = -\frac{\sigma_0}{2\mu_1(1-\kappa_1^2)p}, \quad r < a \tag{14}$$

in which $P_I(s,p)$ is a known function:

$$P_I(s,p) = \frac{1}{s\Delta_I(1-\kappa_1^2)}\{[\tfrac{1}{4}(s^2+\gamma_{21}^2)^2 - s^2\gamma_{11}\gamma_{21}][\delta^{(2)} - \delta^{(3)}e^{-2(\gamma_{11}+\gamma_{21})b}]$$

$$+ s(s^2+\gamma_{21}^2)e^{-(\gamma_{11}+\gamma_{21})b}[\gamma_{21}(\delta^{(1)}\delta^{(4)} - \delta^{(2)}\delta^{(3)}) - \gamma_{11}]$$

$$+ [\tfrac{1}{4}(s^2+\gamma_{21}^2)^2 + s^2\gamma_{11}\gamma_{21}][\delta^{(4)}e^{-2\gamma_{21}b} - \delta^{(1)}e^{-2\gamma_{11}b}]\} \tag{15}$$

The form of A(s,p) that satisfies equations (14) can be found from Copson [6]:

$$A(s,p) = -\sqrt{\frac{2s}{\pi}}\,\frac{\sigma_0 a^{5/2}}{2\mu_1 p(1-\kappa_1^2)}\int_0^1 \sqrt{\xi}\,\Lambda_I^*(\xi,p)\,J_{1/2}(sa\xi)d\xi \tag{16}$$

Here, $J_{1/2}$ is the half order Bessel function of the first kind and $\Lambda_I^*(\xi,p)$ satisfies the Fredholm integral equation

$$\Lambda_I^*(\xi,p) + \int_0^1 \Lambda_I^*(\eta,p)\,M_I(\xi,\eta,p)d\eta = \xi \tag{17}$$

whose kernel

$$M_I(\xi,\eta,p) = \sqrt{\xi\eta}\int_0^\infty s[P_I(\tfrac{s}{a},p) - 1]\,J_{1/2}(s\xi)\,J_{1/2}(s\eta)ds$$

$$= \frac{2}{\pi}\int_0^\infty [P_I(\tfrac{s}{a},p) - 1]\sin(s\xi)\,\sin(s\eta)ds \tag{18}$$

is symmetric in ξ and η. Figures 2 to 4 show the numerical results of equation (17) by varying μ_2/μ_1 and a/b while $\rho_1 = \rho_2$ and $\nu_1 = \nu_2 = 0.29$ are kept the same for all cases. The function $\Lambda_I^*(\xi,p)$ evaluated at the crack border, $\xi = 1$, governs the contribution of the geometric and material parameters on $k_1^*(p)$ which represents the Laplace transform of the stress intensity factor.

Stress intensity factor for normal impact. In order to evaluate $k_1^*(p)$ or $k_1(t)$, the stresses in the matrix layer are first expanded in terms of the local coordinates r_1 and θ_1 for small values of r_1. The local coordinates (r_1,θ_1) are related to (r,θ) in Figure 1 as follows:

$$a + r_1 \cos\theta_1 = r \cos\theta$$

$$r_1 \sin\theta_1 = r \sin\theta \qquad (19)$$

The leading term in the Laplace transform of the local stresses that possess the $1/\sqrt{r_1}$ singularity is

$$k_1^*(p) = \frac{\Lambda_I^*(1,p)}{p} \frac{2}{\pi} \sigma_o \sqrt{a} \qquad (20)$$

Application of the Laplace inversion theorem yields the dynamic stress field around the crack border as a function of time. The result is

$$(\sigma_r)_1(r_1,\theta_1,t) = \frac{k_1(t)}{\sqrt{2r_1}} \cos\frac{\theta_1}{2} (1 - \sin\frac{\theta_1}{2} \sin\frac{3\theta_1}{2}) + O(r_1^o)$$

$$(\sigma_\theta)_1(r_1,\theta_1,t) = \frac{k_1(t)}{\sqrt{2r_1}} 2\nu_1 \cos\frac{\theta_1}{2} + O(r_1^o)$$

$$(\sigma_z)_1(r_1,\theta_1,t) = \frac{k_1(t)}{\sqrt{2r_1}} \cos\frac{\theta_1}{2} (1 + \sin\frac{\theta_1}{2} \sin\frac{3\theta_1}{2}) + O(r_1^o) \qquad (21)$$

$$(\tau_{rz})_1(r_1,\theta_1,t) = \frac{k_1(t)}{\sqrt{2r_1}} \cos\frac{\theta_1}{2} \sin\frac{\theta_1}{2} \cos\frac{3\theta_1}{2} + O(r_1^o)$$

and $k_1(t)$ becomes

$$k_1(t) = \frac{2\sigma_o\sqrt{a}}{\pi} \frac{1}{2\pi i} \int_{Br} \frac{\Lambda_I^*(1,p)}{p} e^{pt} dp \qquad (22)$$

Note that equation (20) is, in fact, the Laplace transform of equation (22). Hence, the functional dependence of r_1 and θ_1 is not affected by the Laplace

transformation and can be evaluated separately. This observation was first made by Sih, Ravera and Embley [7].

Making use of the results for $\Lambda_I^*(1,p)$ in Figures 2 to 4, $k_1(t)$ in equation (22) can be found as given in Figures 5 to 7. The dynamic stress intensity factors $k_1(t)$ for the penny-shaped crack exhibit an oscillatory behavior rising quickly to a peak. As time increases, all curves approach the static value of $k_1 = 2\sigma_0\sqrt{a}/\pi$ [4]. For a crack diameter to layer thickness ratio of $a/b = 1$, the peaks of the $k_1(t)$ curve are sensitive to changes in the shear moduli ratio μ_2/μ_1. Figure 5 indicates that $k_1(t)$ tends to decrease in amplitude as μ_2/μ_1 is reduced from 0.1 to 10.0. The influence of the composite interface on $k_1(t)$ is exhibited in Figures 6 to 7. When the shear modulus of the surrounding material μ_2 is much smaller than the matrix layer with μ_1, the dynamic crack border stress intensity increases as the crack diameter becomes large in comparison with the layer thickness. This effect is clearly evidenced in Figure 6. As expected, $k_1(t)$ increases with decreasing a/b when the shear modulus of the cracked layer is made smaller than the surrounding material, i.e., $\mu_1 < \mu_2$ as illustrated in Figure 7. The result of Embley and Sih [8] is recovered for the homogeneous case, $\mu_1 = \mu_2$.

RADIAL IMPACT

If the penny-shaped crack is sheared uniformly in the radial direction such that axial symmetry is preserved, then $\phi_j^*(r,z,p)$ and $\psi_j^*(r,z,p)$ in equations (8) and (9) remain valid. Let this shear of magnitude τ_0 be applied suddenly and hence the surface tractions, $-\tau_0 H(t)$, are to be specified for $0 \leq r < a$ with $H(t)$ being the Heaviside unit step function. Laplace transform of the conditions on the plane $z = 0$ thus become

$$
\begin{aligned}
&(\tau^*_{rz})_1(r,o,p) = -\frac{\tau_o}{p};\ (\sigma^*_z)_1(r,o,p) = 0,\ 0\le r<a \\
&(u^*_r)_1(r,o,p) = 0;\ (\sigma^*_z)_1(r,o,p) = 0,\ r \ge a
\end{aligned}
\tag{23}
$$

Continuity of the stresses across the interface z = b is satisfied if

$$
\begin{aligned}
&(\sigma^*_z)_1(r,b,p) = (\sigma^*_z)_2(r,b,p) \\
&(\sigma^*_{rz})_1(r,b,p) = (\sigma^*_{rz})_2(r,b,p)
\end{aligned}
\tag{24}
$$

and the same requirement is imposed on the displacements:

$$
\begin{aligned}
&(u^*_r)_1(r,b,p) = (u^*_r)_2(r,b,p) \\
&(u^*_z)_1(r,b,p) = (u^*_z)_2(r,b,p)
\end{aligned}
\tag{25}
$$

Integral equations. As in the case of normal impact, the six unknown functions $A^{(1)}(s,p)$, $A^{(2)}(s,p),\ldots,$ $C^{(2)}(s,p)$ in equations (8) and (9) can be expressed in terms of a single unknown B(s,p). Refer to equations (A.5) in the Appendix. Hence, equations (24) and (25) are satisfied. The remaining boundary conditions in equations (23) are employed to obtain the system of dual integral equations

$$\int_0^\infty B(s,p)\, J_1(rs)ds = 0, \; r \geq a$$

$$\int_0^\infty sP_{II}(s,p)\, B(s,p)\, J_1(rs)ds = -\frac{\tau_0}{2\mu_1(1-\kappa_1^2)p}, \; r < a \tag{26}$$

in which

$$P_{II}(s,p) = \frac{\Delta_I}{\Delta_{II}} P_I(s,p) \tag{27}$$

where $P_I(s,p)$ is already known through equation (15) while $\Delta_I(s,p)$ and $\Delta_{II}(s,p)$ are given by equations (A.2) and (A.6), respectively.

Solving for B(s,p) [6], it can be shown that

$$B(s,p) = -\sqrt{\frac{\pi s}{2}}\, \frac{\tau_0 a^{5/2}}{4\mu_1 p(1-\kappa_1^2)} \int_0^1 \sqrt{\xi}\, \Lambda_{II}^*(\xi,p)\, J_{3/2}(sa\xi)d\xi \tag{28}$$

and $\Lambda_{II}^*(\xi,p)$ satisfies the Fredholm integral equation of the second kind:

$$\Lambda_{II}^*(\xi,p) + \int_0^1 \Lambda_{II}^*(\eta,p)\, M_{II}(\xi,\eta,p)d\eta = \xi \tag{29}$$

whose kernel takes the form

$$M_{II}(\xi,\eta,p) = \sqrt{\xi\eta} \int_0^\infty s[P_{II}(\tfrac{s}{a}, p) - 1]\, J_{3/2}(s\xi)\, J_{3/2}(s\eta)ds \tag{30}$$

Plots of $\Lambda_{II}^*(1,p)$ as a function of c_{21}/pa are shown in Figures 8 to 10 for different values of μ_2/μ_1 and a/h. The curves show that $\Lambda_{II}^*(1,p)$ rises rapidly at first and then levels off.

Stress intensity factor for radial impact. The dynamic crack border stress field corresponding to radial shear can be obtained in the same way and expressed in terms of the coordinates (r_1,θ_1) in equations (19):

$$
\begin{aligned}
(\sigma_r)_1(r_1,\theta_1,t) &= \frac{k_2(t)}{\sqrt{2r_1}} \sin\frac{\theta_1}{2}\left(2+\cos\frac{\theta_1}{2}\cos\frac{3\theta_1}{2}\right) + O(r_1^\circ) \\
(\sigma_\theta)_1(r_1,\theta_1,t) &= \frac{k_2(t)}{\sqrt{2r_1}}\, 2\nu_1 \sin\frac{\theta_1}{2} + O(r_1^\circ) \\
(\sigma_z)_1(r_1,\theta_1,t) &= -\frac{k_2(t)}{\sqrt{2r_1}} \sin\frac{\theta_1}{2}\cos\frac{\theta_1}{2}\cos\frac{3\theta_1}{2} + O(r_1^\circ) \\
(\tau_{rz})_1(r_1,\theta_1,t) &= \frac{k_2(t)}{\sqrt{2r_1}} \cos\frac{\theta_1}{2}\left(1-\sin\frac{\theta_1}{2}\sin\frac{3\theta_1}{2}\right) + O(r_1^\circ)
\end{aligned}
\tag{31}
$$

Note that $k_2(t)$ can be evaluated from

$$
k_2(t) = \frac{\tau_0\sqrt{a}}{4\pi i}\int_{Br} \frac{\overset{*}{\Lambda}_{II}(1,p)}{p}\, e^{pt}dp \tag{32}
$$

once $\overset{*}{\Lambda}_{II}(1,p)$ as given by Figures 8 to 10 is known.

The numerical results in Figures 11 to 13 for $k_2(t)$ as a function of time refer to $\rho_1 = \rho_2$ and $\nu_1 = \nu_2 = 0.29$. The curve with $\mu_1 = \mu_2$ is the solution for the homogeneous material treated previously by Embley and Sih [8]. In general, $k_2(t)$ oscillates with time and can be greater or smaller than the corresponding homogeneous solution depending on whether $\mu_2/\mu_1 < 1$ or $\mu_2/\mu_1 > 1$. Figure 11 displays the variations of $k_2(t)$ for different values of μ_2/μ_1 while a/b is fixed at unity. The influence of the ratio of crack size with layer thickness

is exhibited in Figures 12 and 13 for $\mu_2/\mu_1 = 0.1$ and $\mu_2/\mu_1 = 10.0$, respectively. These two cases show the opposite effect which is to be expected.

CONCLUDING REMARKS

The previous discussion has shown that the dynamic stress intensity factors for an embedded crack can be evaluated analytically by a method similar to that developed for a through crack [1]. An important consideration is to compare the results for these two crack configurations and to draw some general conclusions. First of all, the $k_1(t)$ or $k_2(t)$ factor for the penny-shaped crack tends to rise more quickly than the through crack, i.e., the peak value of $k_1(t)$ or $k_2(t)$ is reached within a shorter period of time. This is because waves emanating from the neighboring points on the periphery of the penny-shaped crack interfere with each other much earlier as compared to a line (or plane) crack where the waves must travel from one end to the other before interference can take place. In general, the maximum value of $k_1(t)$ or $k_2(t)$ for an embedded crack is lower than that for a through crack. For example, Figure 5 gives a peak value of approximately 1.6 for $\pi k_1(t)/2\sigma_0\sqrt{a}$ which corresponds to $a/b = 1.0$ and $\mu_2/\mu_1 = 0.1$. This occurs at $c_{21}t/a \simeq 1.6$ and yields $k_1(t) \simeq 1.02\ \sigma_0\sqrt{a}$. The corresponding case of a through crack [1] renders $k_1(t) \simeq 2.40\ \sigma_0\sqrt{a}$ and $c_{21}t/a \simeq 3.0$. The difference in $k_1(t)$ is more than a factor of two and is more pronounced as the ratio a/b is increased. For embedded cracks that are non-circular in shape, approximate estimates of $k_1(t)$ can be made by taking the solution for the through crack as an upper limit and that of the circular crack as a lower limit.

In the absence of axisymmetry, the dynamic stress analysis will become exceedingly difficult and it will be more feasible to solve the crack problem numerically. In such cases, the solutions obtained here can be used to guide the development of numerical procedures.

APPENDIX: EXPRESSIONS FOR $A^{(i)}(s,p),\ldots,\ C^{(i)}(s,p)$

Normal impact. The functions $A^{(1)}(s,p)$, $A^{(2)}(s,p),\ldots,\ C^{(2)}(s,p)$ for the wave potentials in equations (8) and (9) can be expressed in terms of a single unknown $A(s,p)$ for normal impact

$$A^{(1)}(s,p) = [\tfrac{1}{2}\,(s^2+\gamma_{21}^2)(\delta^{(2)} + \delta^{(4)}e^{-2\gamma_{21}b}) - s\gamma_{11}e^{-(\gamma_{11}+\gamma_{21})b}]\,\frac{A(s,p)}{\Delta_I}$$

$$A^{(2)}(s,p) = -\,[s\gamma_{11}e^{-(\gamma_{11}+\gamma_{21})b} + \tfrac{1}{2}\,(s^2+\gamma_{21}^2)e^{-2\gamma_{11}b}\,(\delta^{(1)} + \delta^{(3)}e^{-2\gamma_{21}b})]$$

$$\times\,\frac{A(s,p)}{\Delta_I}$$

$$B^{(1)}(s,p) = -\,[\delta^{(1)}A^{(1)}e^{-\gamma_{11}b} + \delta^{(2)}A^{(2)}e^{\gamma_{11}b}]$$

$$B^{(2)}(s,p) = -\,[\delta^{(3)}A^{(1)}e^{-\gamma_{11}b} + \delta^{(4)}A^{(2)}e^{\gamma_{11}b}] \tag{A.1}$$

$$C^{(1)}(s,p) = \frac{e^{\gamma_{12}b}}{s^2-\gamma_{12}\gamma_{22}}\,[(s^2-\gamma_{11}\gamma_{22})A^{(1)}e^{-\gamma_{11}b} + (s^2+\gamma_{11}\gamma_{22})A^{(2)}e^{\gamma_{11}b}$$

$$-\ s(\gamma_{21}-\gamma_{22})B^{(1)}e^{-\gamma_{21}b} + s(\gamma_{21}+\gamma_{22})B^{(2)}e^{\gamma_{21}b}]$$

$$C^{(2)}(s,p) = \frac{e^{\gamma_{22}b}}{s^2-\gamma_{12}\gamma_{22}}\,[s(\gamma_{12}-\gamma_{11})A^{(1)}e^{-\gamma_{11}b} + s(\gamma_{11}+\gamma_{12})e^{\gamma_{11}b}$$

$$+\ (s^2-\gamma_{21}\gamma_{12})B^{(1)}e^{-\gamma_{21}b} + (s^2+\gamma_{21}\gamma_{12})B^{(2)}e^{\gamma_{21}b}]$$

in which Δ_I stands for

$$\Delta_I(s,p) = \frac{p^2}{2c_{21}^2}\gamma_{11}\,[\delta^{(2)} + \delta^{(3)}e^{-2(\gamma_{11}+\gamma_{21})b} + \delta^{(4)}e^{-2\gamma_{21}b}$$

$$+ \delta^{(1)}e^{-2\gamma_{11}b}] \qquad \text{(A.2)}$$

and $\delta^{(1)}, \delta^{(2)},\ldots, \delta^{(4)}$ are further expressed in terms of $e^{(1)}, e^{(2)},\ldots, e^{(8)}$ as the following:

$$\delta^{(1)}(s,p) = (e^{(1)}e^{(6)} - e^{(2)}e^{(7)})/(e^{(1)}e^{(6)} - e^{(2)}e^{(5)})$$

$$\delta^{(2)}(s,p) = (e^{(4)}e^{(6)} - e^{(2)}e^{(8)})/(e^{(1)}e^{(6)} - e^{(2)}e^{(5)})$$

$$\text{(A.3)}$$

$$\delta^{(3)}(s,p) = (e^{(1)}e^{(7)} - e^{(3)}e^{(5)})/(e^{(1)}e^{(6)} - e^{(2)}e^{(5)})$$

$$\delta^{(4)}(s,p) = (e^{(1)}e^{(8)} - e^{(4)}e^{(5)})/(e^{(1)}e^{(6)} - e^{(2)}e^{(5)})$$

The quantities in equations (A.3) are complicated functions of the materials parameters and transform variables. They are given by

$$e^{(1)}(s,p) = -\,s\gamma_{21} + \frac{s\mu_2}{\mu_1(s^2-\gamma_{12}\gamma_{22})}[\tfrac{1}{2}(\gamma_{21}-\gamma_{22})(s^2+\gamma_{22}^2) + \gamma_{22}(s^2-\gamma_{21}\gamma_{12})]$$

$$e^{(2)}(s,p) = s\gamma_{21} - \frac{s\mu_2}{\mu_1(s^2-\gamma_{12}\gamma_{22})}[\tfrac{1}{2}(\gamma_{21}+\gamma_{22})(s^2+\gamma_{22}^2) - \gamma_{22}(s^2+\gamma_{21}\gamma_{12})]$$

$$e^{(3)}(s,p) = \tfrac{1}{2}(s^2+\gamma_{21}^2) - \frac{\mu_2}{\mu_1(s^2-\gamma_{12}\gamma_{22})}[\tfrac{1}{2}(s^2+\gamma_{22}^2)(s^2-\gamma_{11}\gamma_{22})$$

$$+ s^2\gamma_{22}(\gamma_{11}-\gamma_{12})]$$

$$e^{(4)}(s,p) = \frac{1}{2}(s^2+\gamma_{21}^2) - \frac{\mu_2}{\mu_1(s^2-\gamma_{12}\gamma_{22})}[\frac{1}{2}(s^2+\gamma_{22}^2)(s^2+\gamma_{11}\gamma_{22})$$

$$- s^2\gamma_{22}(\gamma_{11}+\gamma_{12})]$$

$$e^{(5)}(s,p) = -\frac{1}{2}(s^2+\gamma_{21}^2) + \frac{\mu_2}{\mu_1(s^2-\gamma_{12}\gamma_{22})}[s^2\gamma_{12}(\gamma_{21}-\gamma_{22})$$

$$+ \frac{1}{2}(s^2+\gamma_{22}^2)(s^2-\gamma_{21}\gamma_{12})]$$

(A.4)

$$e^{(6)}(s,p) = -\frac{1}{2}(s^2+\gamma_{21}^2) - \frac{\mu_2}{\mu_1(s^2-\gamma_{12}\gamma_{22})}[s^2\gamma_{12}(\gamma_{21}+\gamma_{22})$$

$$- \frac{1}{2}(s^2+\gamma_{22}^2)(s^2+\gamma_{21}\gamma_{12})]$$

$$e^{(7)}(s,p) = s\gamma_{11} - \frac{s\mu_2}{\mu_1(s^2-\gamma_{12}\gamma_{22})}[\gamma_{12}(s^2-\gamma_{11}\gamma_{22}) + \frac{1}{2}(s^2+\gamma_{22}^2)(\gamma_{11}-\gamma_{12})]$$

$$e^{(8)}(s,p) = -s\gamma_{11} - \frac{s\mu_2}{\mu_1(s^2-\gamma_{12}\gamma_{22})}[\gamma_{12}(s^2+\gamma_{11}\gamma_{22}) - \frac{1}{2}(s^2+\gamma_{22}^2)(\gamma_{11}+\gamma_{12})]$$

Radial impact. For radial impact, $A^{(1)}(s,p)$, $A^{(2)}(s,p),\ldots,$ $C^{(2)}(s,p)$ in equations (8) and (9) can be expressed in terms of $B(s,p)$ as

$$A^{(1)}(s,p) = -[s\gamma_{21}(\delta^{(2)} - \delta^{(4)}e^{-2\gamma_{21}b}) + \frac{1}{2}(s^2+\gamma_{21}^2)e^{-(\gamma_{11}+\gamma_{21})b}]\frac{B(s,p)}{\Delta_{II}}$$

(A.5)

$$A^{(2)}(s,p) = [s\gamma_{21}e^{-2\gamma_{11}b}(\delta^{(1)} - \delta^{(3)}e^{-2\gamma_{21}b}) + \frac{1}{2}(s^2+\gamma_{21}^2)e^{-(\gamma_{11}+\gamma_{21})b}]$$

$$\times \frac{B(s,p)}{\Delta_{II}}$$

where

$$\Delta_{II} = \frac{p^2}{2c_{21}^2} \gamma_{21} [\delta^{(2)} + \delta^{(3)} e^{-2(\gamma_{11}+\gamma_{21})b} - \delta^{(4)} e^{-2\gamma_{21}b}$$

$$- \delta^{(1)} e^{-2\gamma_{11}b}] \tag{A.6}$$

The remaining functions $B^{(1)}(s,p)$, $B^{(2)}(s,p)$, etc., can be related to $B(s,p)$ through $A^{(1)}(s,p)$ and $A^{(2)}(s,p)$ since the last four expressions in equations (A.1) for normal impact also apply to radial impact.

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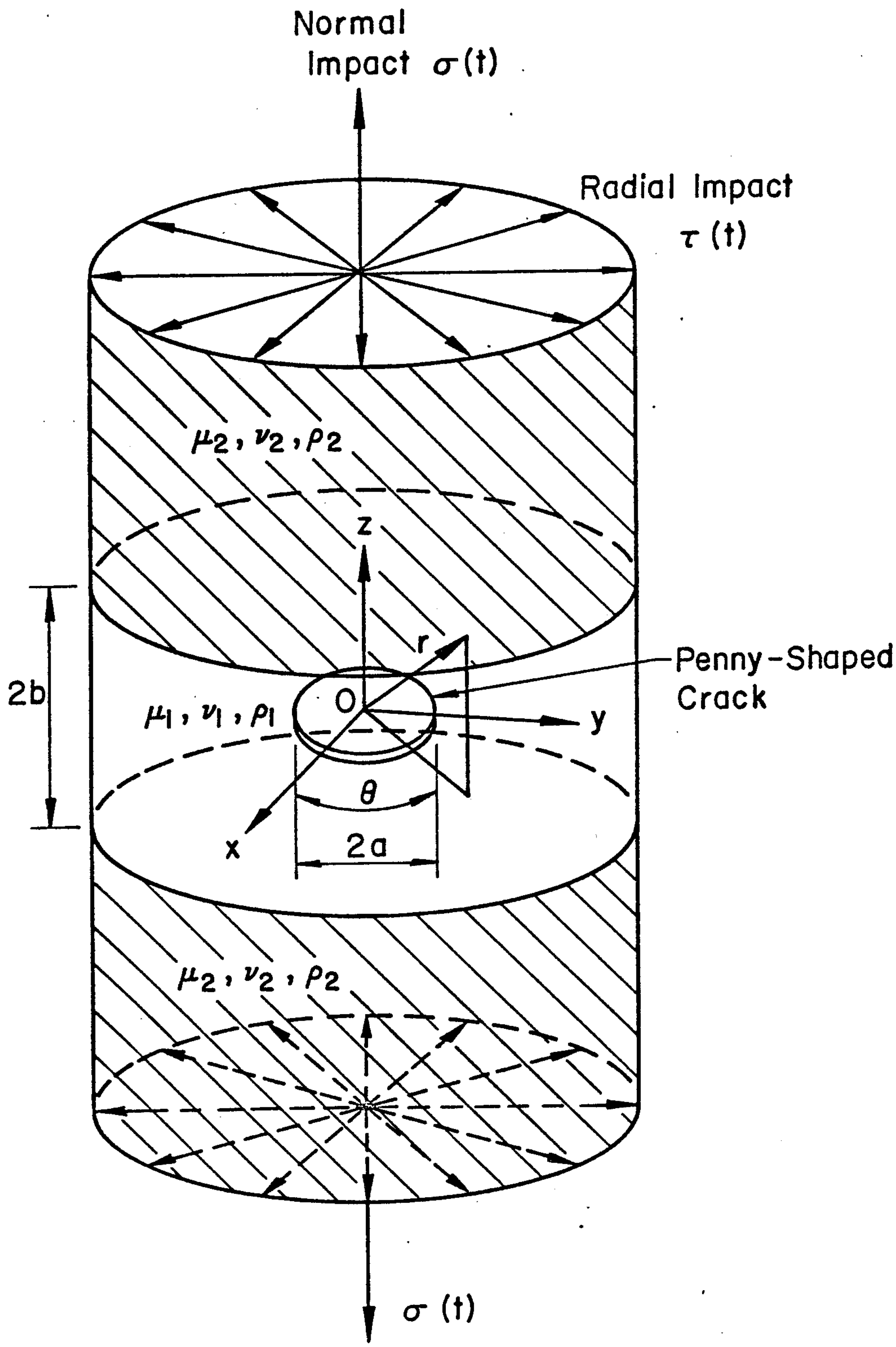

Figure 1 - Penny-shaped crack embedded in a matrix layer under normal and radial impact

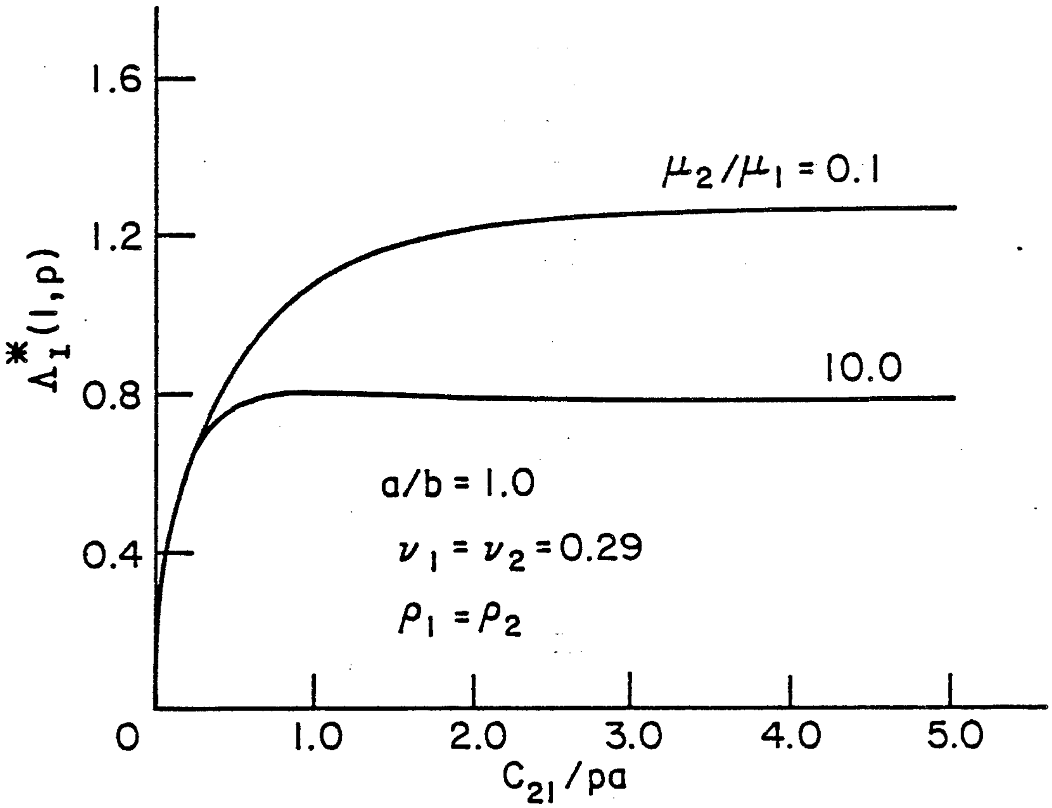

Figure 2 - Plot of $\Lambda_I^*(1,p)$ versus c_{21}/pa for a/b = 1.0

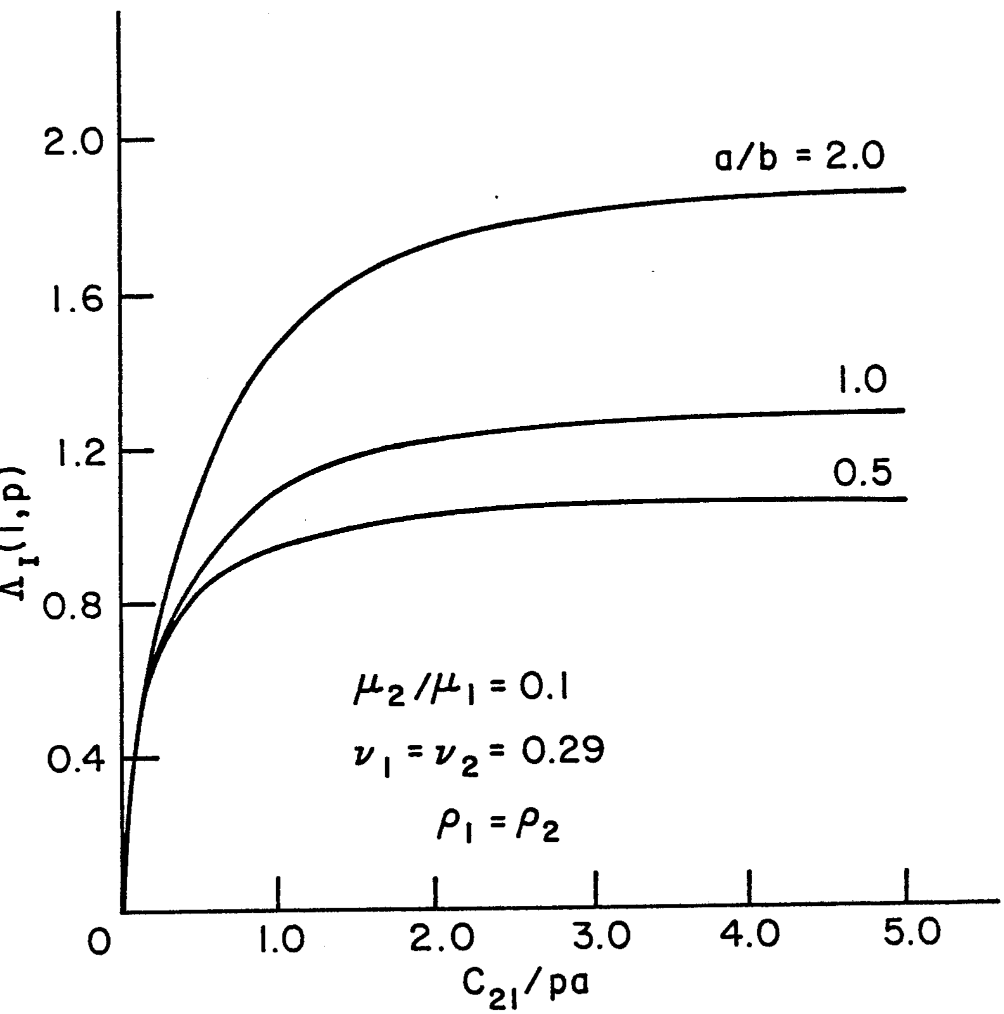

Figure 3 - Plot of $\Lambda_I^*(1,p)$ versus c_{21}/pa for $\mu_2/\mu_1 = 0.1$

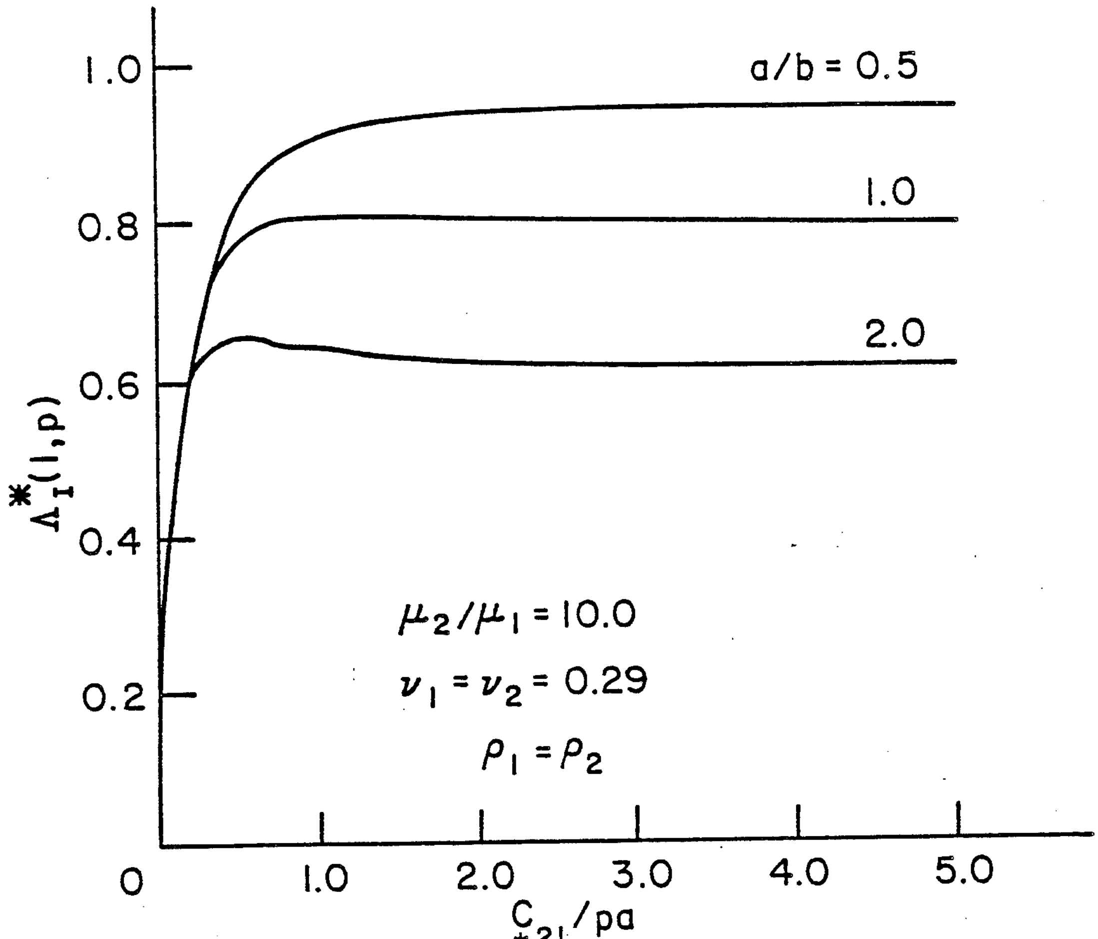

Figure 4 - Plot of $\Lambda_I^*(1,p)$ versus c_{21}/pa for $\mu_2/\mu_1 = 10.0$

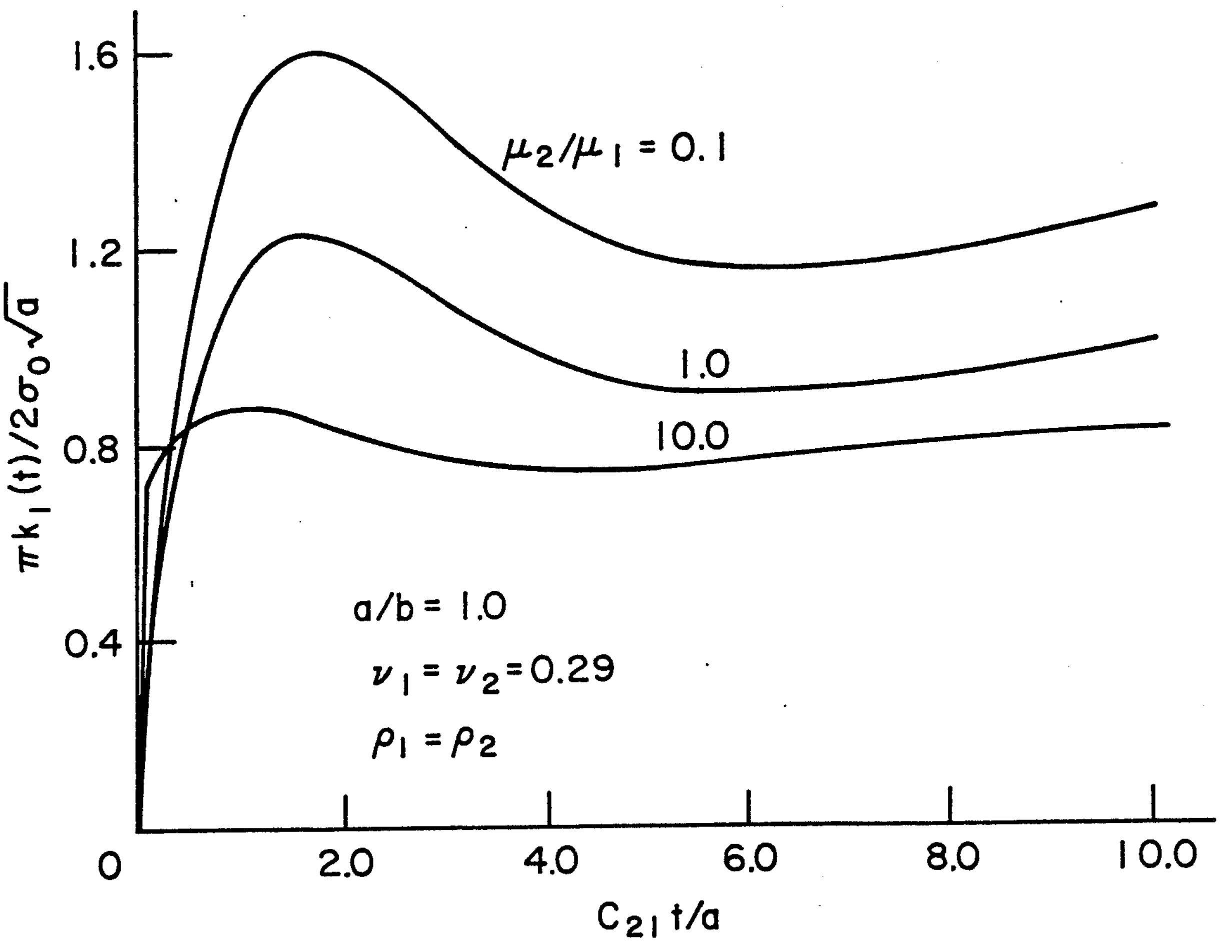

Figure 5 - Dynamic stress intensity factor $k_1(t)$ for penny-shaped crack with $a/b = 1.0$

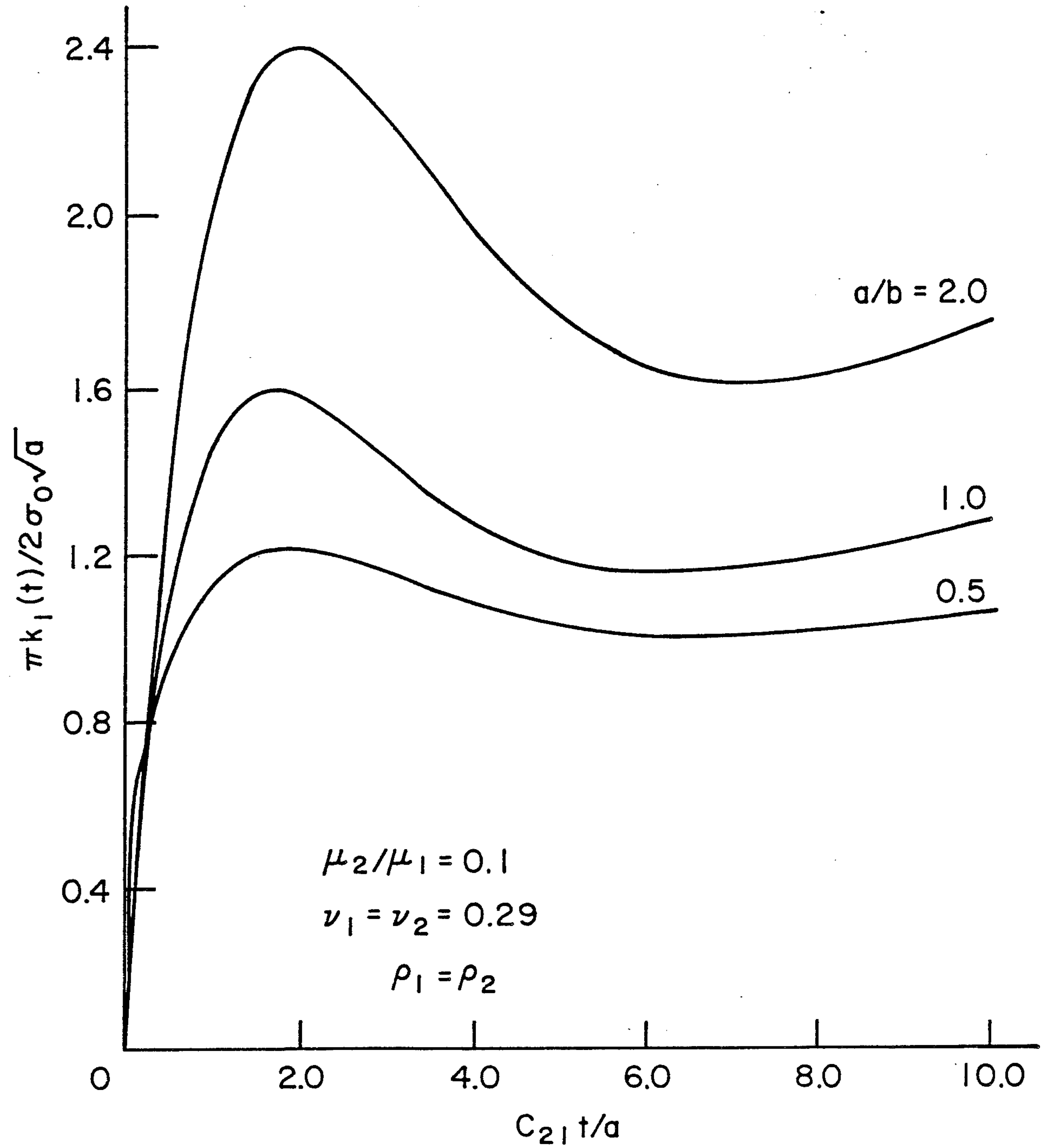

Figure 6 - Dynamic stress intensity factor $k_1(t)$ for penny-shaped crack with $\mu_2/\mu_1 = 0.1$

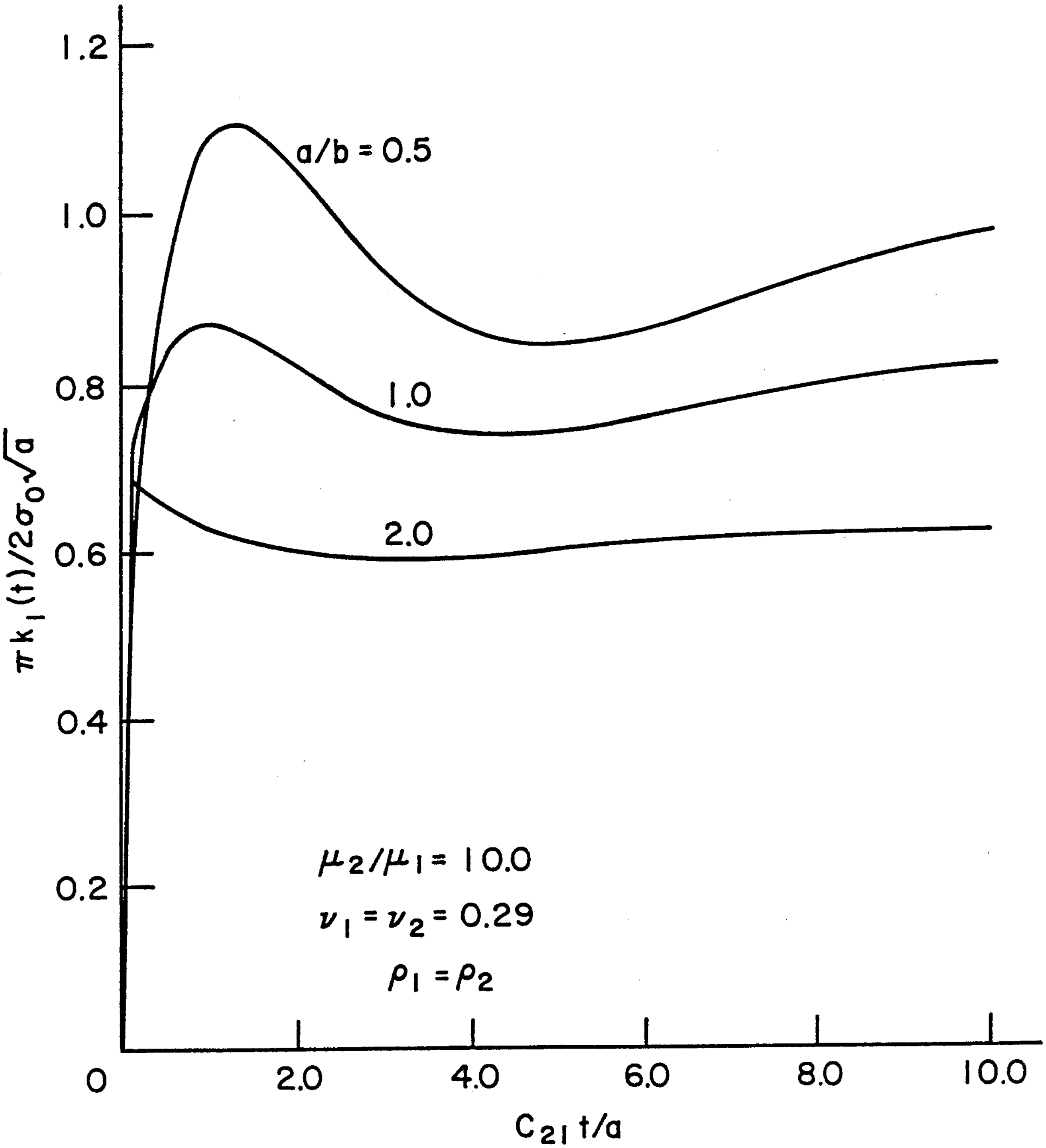

Figure 7 - Dynamic stress intensity factor $k_1(t)$ for penny-shaped crack with $\mu_2/\mu_1 = 10.0$

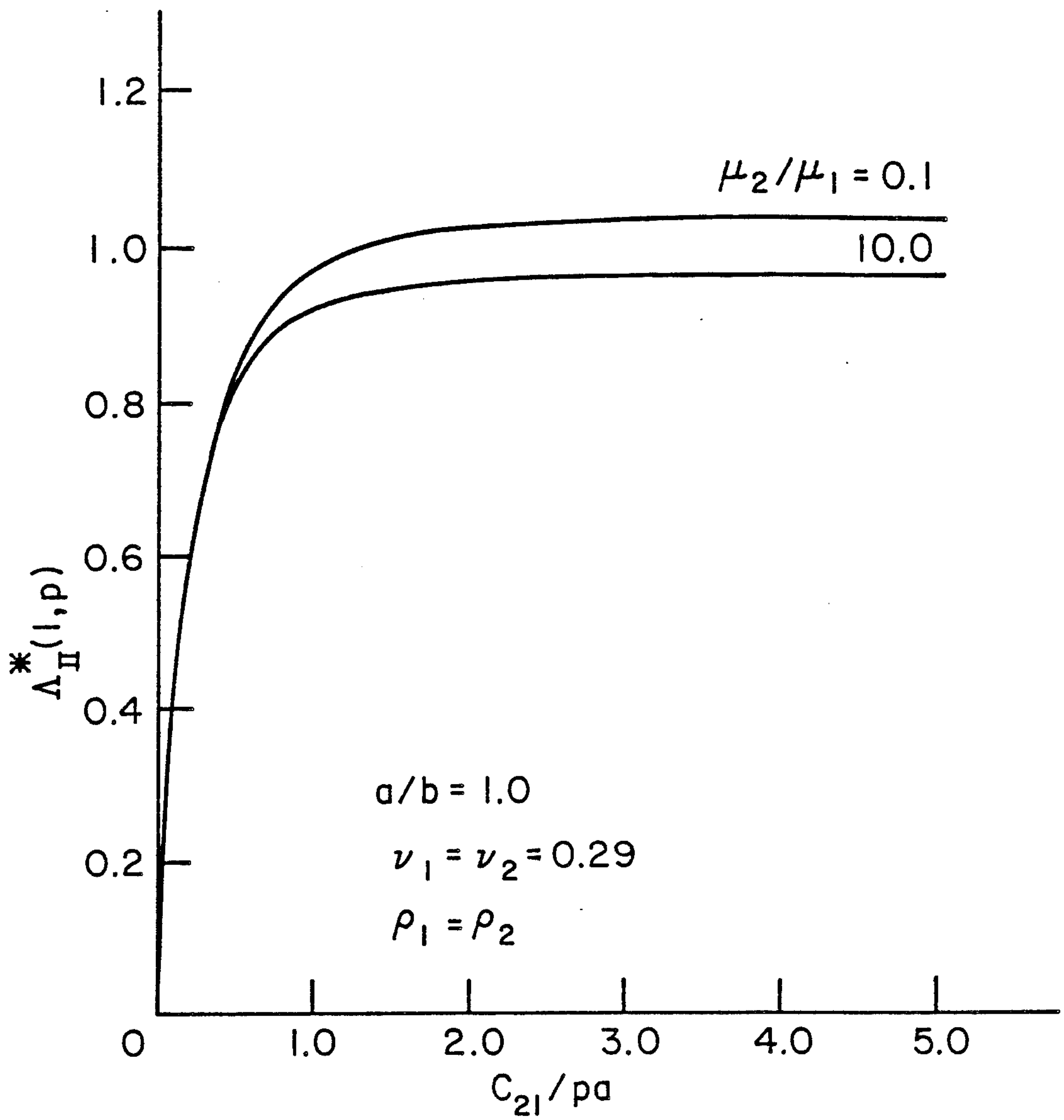

Figure 8 - Variations of $\Lambda^*_{II}(1,p)$ with c_{21}/pa for a/b = 1.0

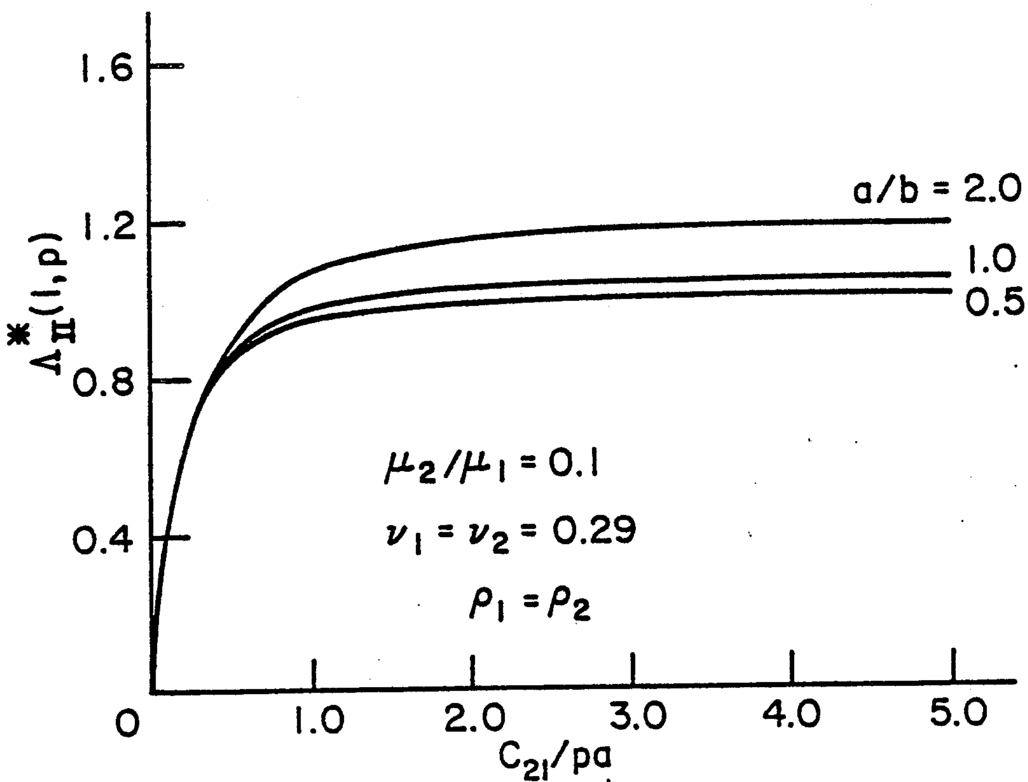

Figure 9 - Variations of $\Lambda^*_{II}(1,p)$ with c_{21}/pa for $\mu_2/\mu_1 = 0.1$ and varying a/b

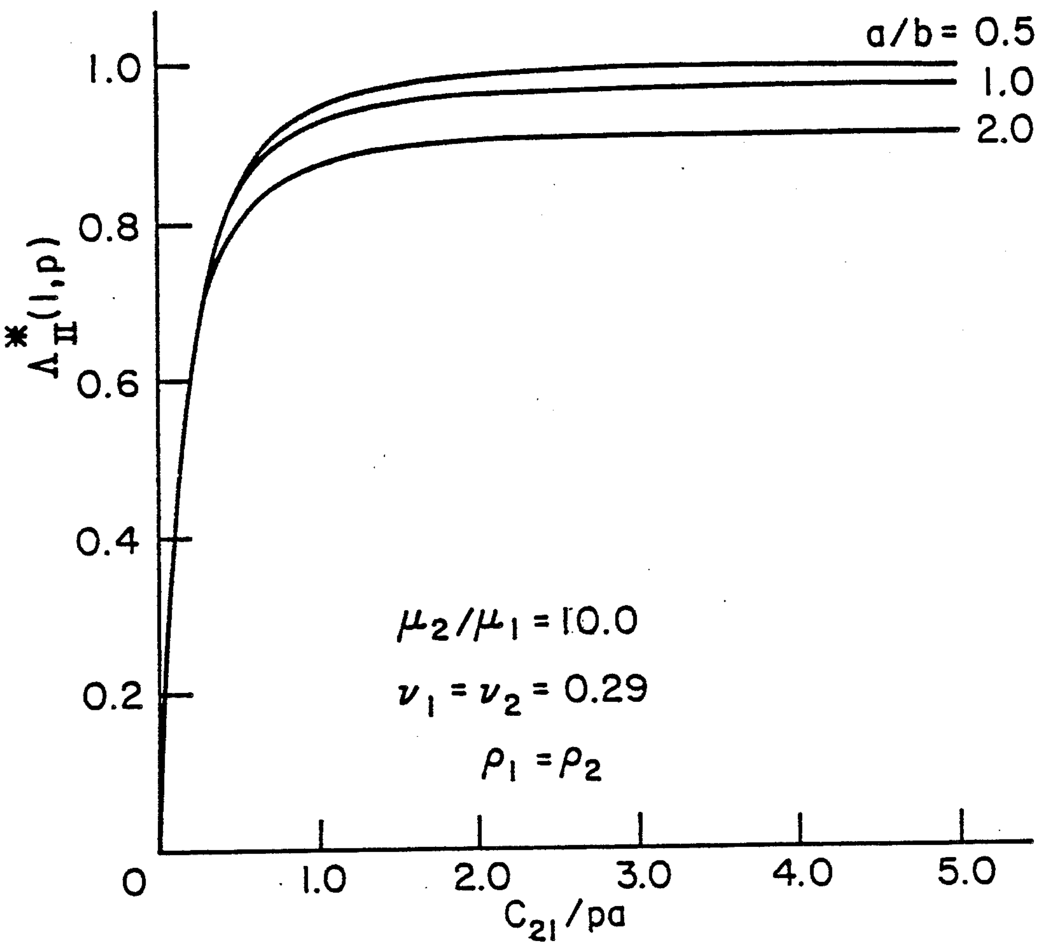

Figure 10 - Variations of $\Lambda^*_{II}(1,p)$ with c_{21}/pa for $\mu_2/\mu_1 = 10$ and varying a/b

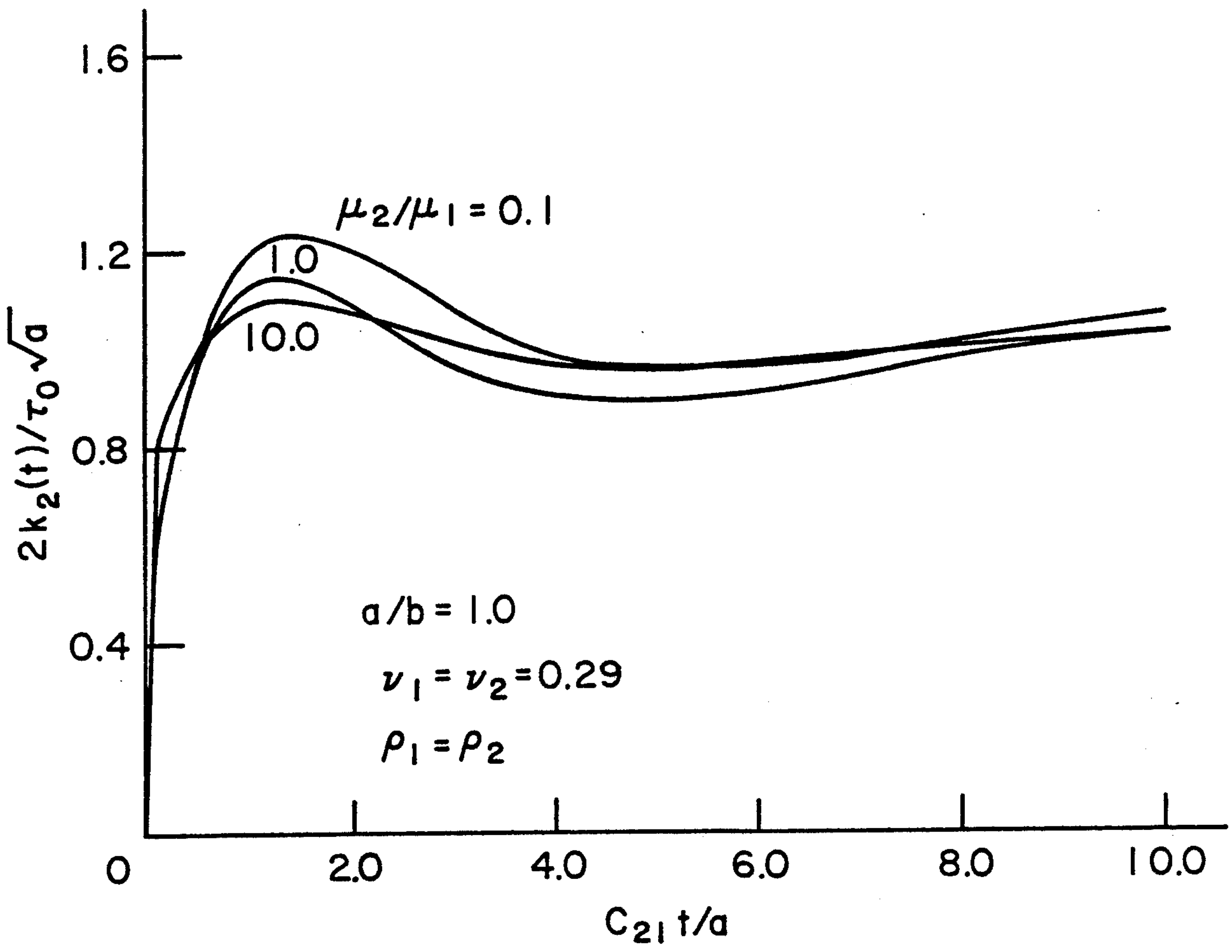

Figure 11 - Stress intensity factor $k_2(t)$ versus time for a penny-shaped crack with a/b = 1.0

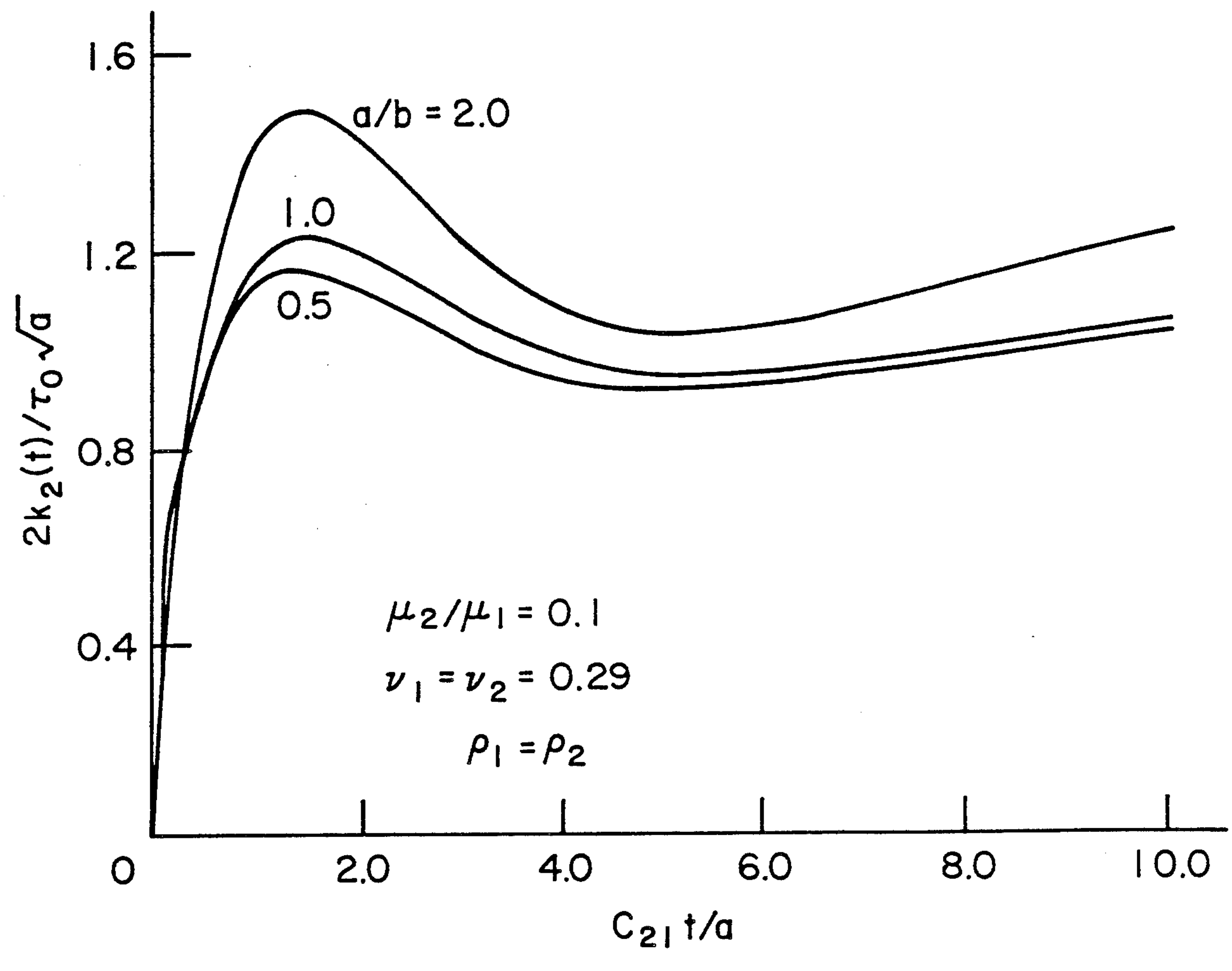

Figure 12 - Stress intensity factor $k_2(t)$ versus time for a penny-shaped crack with $\mu_2/\mu_1 = 0.1$

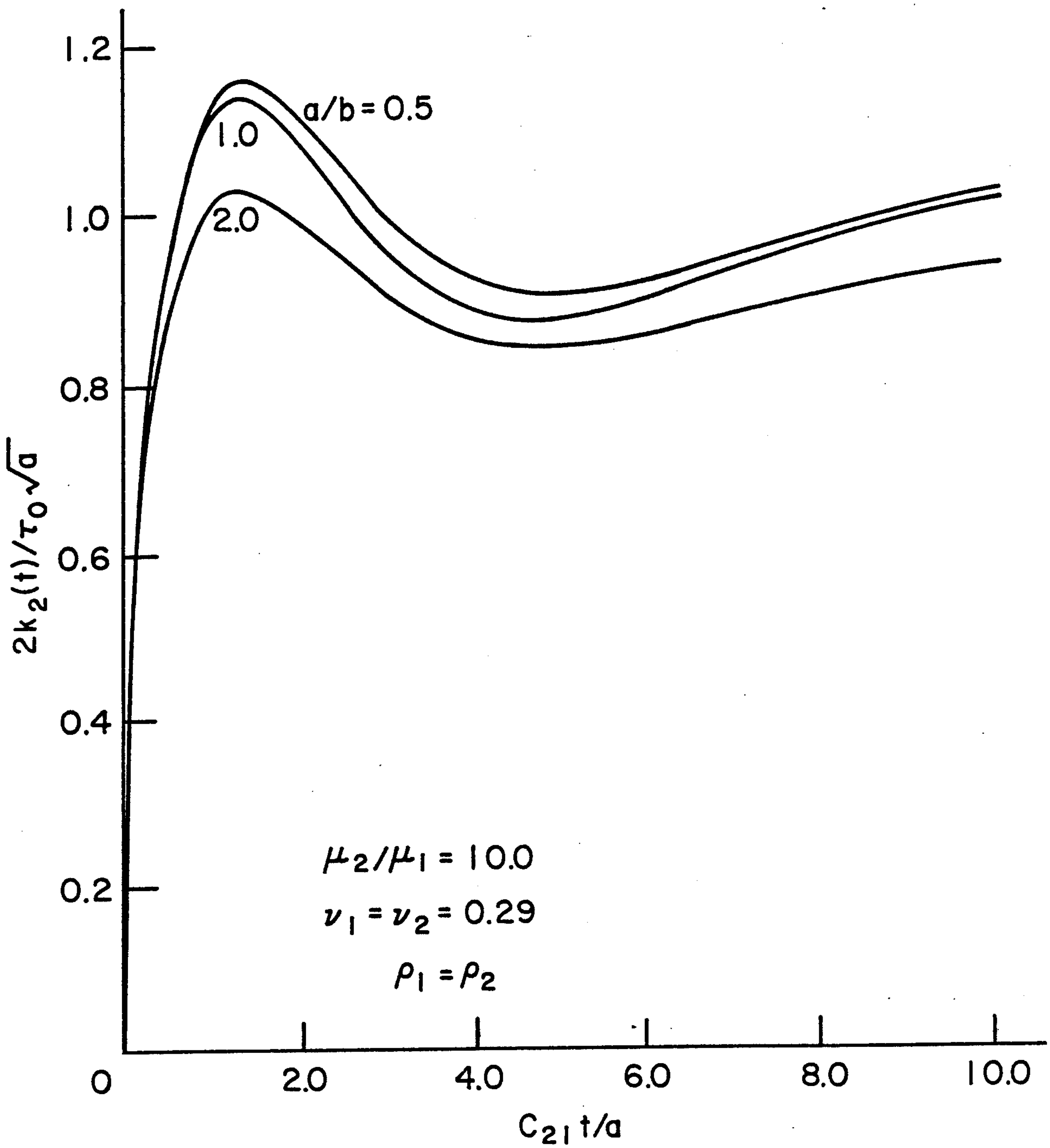

Figure 13 - Stress intensity factor $k_2(t)$ versus time for a penny-shaped crack with $\mu_2/\mu_1 = 10.0$

Axial impact

```
      PROGRAM BETA(INPUT,OUTPUT,PUNCH,PLOT,TAPE 99=PLOT)
      REAL NON(4),F(4,4,1),G(4,4),D(4),PT(4)
      REAL B(4),C(4)
      REAL LP(50),DTA(50)
      EQUIVALENCE (NON,B)
       COMMON K1,K2,K3,K4
      COMMON/AUX/H,P,PK1,PK2,BMU,X,Y
      LP(1)=0.0
      DTA(1)=0.0
      READ 2,K1,K2,K3,K4
    2 FORMAT(I2)
*   K1 = ORDER OF SYSTEM OF EQUATIONS
*   K2 = NO. OF DISTINCT KERNELS
*   K3 = NO. OF DATA POINTS
*   K4 = NO. OF DATA SETS TO BE EVALUATED
*   SET UP DATA POINTS
      AK=K3
      DO 5 N=1,K3
      AN=N
    5 PT(N)=AN/AK
*   SET UP INTEGRATION MATRIX
      M=K3-2
      N=K3-1
      A=K3
      A=1./(3.*A)
      DO 10 K=2,M,2
   10 D(K)=2.*A
      DO 15 K=1,N,2
   15 D(K)=4.*A
      D(K3)=A
*   CALCULATE NONHOMOGENEOUS TERMS
      RHS=1.0
      DO 22 I=1,K2
      PRINT 9
    9 FORMAT(1H1)
      READ 61,BMU
   61 FORMAT(F10.5)
      DO 999 II=1,K4
      DO 35 N=1,K3
   35 NON(N)=RHS*PT(N)
*   CALCULATE KERNEL MATRICES
      CALL CONST(I)
      DO 20 N=1,K3
      DO 20 M=1,K3
      IF(M-N)25,30,30
   25 F(M,N,I)=F(N,M,I)
      GO TO 20
   30 F(M,N,I)=FU(I,PT(M),PT(N))
   20 CONTINUE
      CALL CHANGE(F,G,D,I)
      CALL LINEQ(G,B,C,    K3)
       DO 40 L=1,K3
      PRINT 6,PT(L),NON(L)
    6 FORMAT(5X,F8.4,F15.6)
   40 CONTINUE
      LP(II+1)=NON(K3)
      DTA(II+1)=P
  999 CONTINUE
      PUNCH 66,(DTA(IX),LP(IX),IX=1,19)
   66 FORMAT(2F10.5)
      CALL LAPINV(DTA,LP)
   22 CONTINUE
      END


      FUNCTION SIMP(I,A,B)
      COMMON/AUX/H,P,PK1,PK2,BMU,X,Y
      DEL=0.25*(B-A)
      IF(DEL)40,45,50
   45 SIMP=0.0
      RETURN
   50 CONTINUE
      SA=Z(I,A)+Z(I,B)
      SB=Z(I,A+2.*DEL)
      SC=Z(I,A+DEL)+Z(I,A+3.*DEL)
```

```
      S1=(DEL/3.)*(SA+2.*SB+4.*SC)
      IF(S1.EG.0.0) GO TO 45
      K=8
35    S3=SB+SC
      DEL=0.5*DEL
      SC=Z(I,A+DEL)
      J=K-1
      DO 5 N=3,J,2
      AN=N
5     SC=SC+Z(I,A+AN*DEL)
      S2=(DEL/3.)*(SA+2.*SB+4.*SC)
      DIF=ABS((S2-S1)/S1)
      ER=0.01
      IF(DIF-ER) 30,25,25
30    SIMP=S2
      RETURN
25    K=2*K
      S1=S2
      IF(K-2048) 35,35,40
40    PRINT 42,I,A,B
 42   FORMAT(5X,* INT. DOES NOT CONVERGE *,I3,2F9.4)
      PRINT 60,X,Y
60    FORMAT(2F10.5)
      DO 70 J=1,10
      DIP=J
      DIP=DIP/10.
      W=Z(I,DIF)
      PRINT 60,W
70    CONTINUE
      CALL EXIT
      END


      SUBROUTINE CHANGE(F,G,D,I)
      REAL F(4,4,1),G(4,4),D(4)
      COMMON K1,K2,K3,K4
      DO 10 N=1,K3
      DO 10 M=1,K3
      G(M,N)   =F(M,N,I)*D(N)
10    CONTINUE
      DO 20 N=1,K3
20    G(N,N)=G(N,N)+1.0
      RETURN
      END


      SUBROUTINE LINEQ(A,B,T,N)
      REAL A(N,N),B(N),T(N)
      DO 5 I=2,N
 5    A(I,1)=A(I,1)/A(1,1)
      DO 10 K=2,N
      M=K-1
      DO 15 I=1,N
15    T(I)=A(I,K)
      DO 20 J=1,M
      A(J,K)=T(J)
      J1=J+1
      DO 20 I=J1,N
      T(I)=T(I)-A(I,J)*A(J,K)
20    CONTINUE
      A(K,K)=T(K)
      IF(K.EQ.N) GO TO 10
      M=K+1
      DO 25 I=M,N
25    A(I,K)=T(I)/A(K,K)
10    CONTINUE
*  BACK SUBSTITUTE
      DO 30 I=1,N
      T(I)=B(I)
      M=I+1
      IF(M.GT.N) GO TO 30
      DO 30 J=M,N
      B(J)=B(J)-A(J,I)*T(I)
30    CONTINUE
      DO 35 I=1,N
```

```
      K=N+1-I
      B(K)=T(K)/A(K,K)
      K1=K-1
      IF(K1.EQ.0) GO TO 35
      DO 35 J1=1,K1
      J=K-J1
      T(J)=T(J)-A(J,K)*B(K)
   35 CONTINUE
      RETURN
      END


      FUNCTION FU(I,A,B)
      COMMON/AUX/H,P,PK1,PK2,BMU,X,Y
      X=A
      Y=B
      IF(A*B)5,10,5
   10 FU=0.0
      RETURN
    5 SUM=SIMP(I,0.0,5.0)
      ER=0.01
      DEL =5.0
   20 UP=DEL+5.0
      ADDL=SIMP(I,DEL,UP)
      DEL =UP
      TEST=ABS(ADDL/SUM)
      SUM=SUM+ADDL
      IF(TEST-ER)15,20,20
   15 FU=SQRT(X*Y)*SUM
      RETURN
      END


      SUBROUTINE CONST(I)
      COMMON/AUX/H,P,PK1,PK2,BMU,X,Y
      PR1=0.29
      PR2=0.29
      PK1=SQRT((1.-2.*PR1)/(2.*(1.-PR1)))
      PK2=SQRT((1.-2.*PR2)/(2.*(1.-PR2)))
      READ 1,P
    1 FORMAT(F10.5)
      HH=0.1
      HH=10.0
      HH=5.0
      HH=4.0
      HH=1.0
      HH=0.5
      HH=2.0
      H=1./HH
      PRINT 2,BMU,PR1,PR2,HH,P
    2 FORMAT(////5X,* MU2/MU1 =*F6.2,* NU1 =*F4.2,* NU2 =*F4.2///5X,* A
     1/H =*F4.2,* C21/PA =*F4.2)
      RETURN
      END


      FUNCTION Z(I,S)
      COMMON/AUX/H,P,PK1,PK2,BMU,X,Y
      BESJH(A)=SQRT(2.*A/PI)*SIN(A)/A
      PI=3.1415926
      IF(S-0.0)5,5,10
    5 Z=0.0
      RETURN
   10 CONTINUE
      PP=P*P
      C1=PK1*PK1
      C2=PK2*PK2
      CC=1.-C1
      GA=SQRT(S*S+C1/PP)
      GB=SQRT(S*S+1./PP)
      GC=SQRT(S*S+C2/BMU/PP)
      GD=SQRT(S*S+1./BMU/PP)
      AA=S*S+1./PP/2.
      AB=1.-BMU
```

```
100           AC=S*S-GC*GD
103           AD=(GB-GD)/AC/PP/2.*BMU
110           AE=(GB+GD)/AC/PP/2.*BMU
115           AF=(S*S-GA*GD)/AC/PP/2.*BMU
123           AG=(S*S+GA*GD)/AC/PP/2.*BMU
131           AH=(S*S-GB*GC)/AC/PP/2.*BMU
137           AI=(S*S+GB*GC)/AC/PP/2.*BMU
145           AJ=(GA-GC)/AC/PP/2.*BMU
152           AK=(GA+GC)/AC/PP/2.*BMU
157           A1=-(AB*GB-AD)
162           A2=AB*GB-AE
164           A3=AA-BMU*S*S-AF
171           A4=AA-BMU*S*S-AG
174           A5=-AA+BMU*S*S+AH
200           A6=-AA+BMU*S*S+AI
203           A7=S*(AE*GA-AJ)
206           A8=-S*(AE*GA-AK)
211           BA=A1*A6-A2*A5
214           BB=A3*A6-S*A2*A7
217           BC=A4*A6-S*A2*A8
222           BD=S*A1*A7-A3*A5
226           BE=S*A1*A8-A4*A5
231           B1=BB/BA
233           B2=BC/BA
235           B3=BD/BA
236           B4=BE/BA
240           EA=2.*GA*H
242           EB=2.*GB*H
244           EC=(EA+EB)/2.
246           ED=2.*EC
247           E1=EXP(-EA)
252           E2=EXP(-EB)
256           E3=EXP(-EC)
262           E4=EXP(-ED)
266           DL=B2+B3*E4+B4*E2+B1*E1
275           D1=2.*PP/CC/GB/DL
301           D2=AA*AA-S*S*GA*GB
307           D3=B2-B3*E4
312           D4=2.*AA*(GB*(B1*B4-B2*B3)-S*S*GA)*E3
321           D5=(AA*AA+S*S*GA*GB)*(B4*E2-B1*E1)
332           F=D1*(D2*D3+D4+D5)
337           Z=(F-S)*BESJH(S*X)*BESJH(S*Y)
347           RETURN
350           END


              SUBROUTINE LAPINV(GLAM,PHI)
        C     THIS PROGRAM EVALUATES THE COEFFICIENTS FOR SERIES
        C     OF JACOBI POLYNOMIALS WHICH REPRESENTS A LAPLACE
        C     INVERSION INTEGRAL
  5           REAL MUL
  5           DIMENSION A(50),GLAM(50),PHI(50),C(4,50)
  5           DIMENSION BK(101),TT(101)
  5           COMMON/2/TI,TF,DT,MN,BK,TT
  5           READ 1,NN,MN,MM
 16         1 FORMAT(3I2)
 16           READ 2,TI,TF,DT
 30         2 FORMAT(3F10.5)
 30           PRINT 99
 34        99 FORMAT(1H1)
 34           CALL SPLICE(GLAM,PHI,MM,C)
 40           PRINT 101
 44       101 FORMAT(/////5X,*   GLAM              PHI     *)
 44           PRINT 102,(GLAM(I),PHI(I),I=1,MM)
 65       102 FORMAT(5X,F10.5,5X,F10.5)
 65           M11=MM-1
 67           PRINT 300
 73       300 FORMAT(////5X,*    C(1,J)            C(2,J)             C(3,J)             C(4
             1,J)  *)
 73           PRINT 103,((C(I,J),I=1,4),J=1,M11)
112       103 FORMAT(5X,F10.5,5X,F10.5,5X,F10.5,5X,F10.5)
112           PRINT 99
116           DO 10 I=1,NN
121           READ 3,BET,DEL
130         3 FORMAT(2F10.5)
130           PRINT 98,BET,DEL
```

```
140        98 FORMAT(/////5X,*BETA =*F5.3,* DELTA =*F5.3)
140           DO 11 L=1,MN
143           AL=L
144           S=1./(AL+BET)/DEL
150           CALL SPLINE(GLAM,PHI,MM,C,S,G)
153           F=G*S
155           IF(AL-2.)81,82,83
161        81 A(1)=(1.+BET)*DEL*F
165           GO TO 11
165        82 A(2)=((2.+BET)*DEL*F-A(1))*(3.+BET)
175           GO TO 11
175        83 CONTINUE
175           TOP=1.
177           L1=L-1
201           AL1=L1
202           DO 12 J=1,L1
203           AJ=J
204           TOP=AJ*TOP
206        12 CONTINUE
210           L2=2*L-1
212           BOT=1.
214           DO 13 J=L,L2
215           AJ=J
216           BOT=(AJ+BET)*BOT
221        13 CONTINUE
223           MUL=BOT/TOP
225           SUM=0.0
226           DO 14 N=1,L1
227           AN=N
230           IF(AN-2.)85,86,87
233        85 TOD=1.
235           GO TO 88
235        86 TOD=AL1
237           GO TO 88
237        87 CONTINUE
237           TOD=1.
241           ICH=L1-(N-2)
244           DO 15 J=ICH,L1
245           AJ=J
246           TOD=AJ*TOD
250        15 CONTINUE
252        88 CONTINUE
252           BOD=1.
254           JA=L1+N
256           DO 16 J=L,JA
260           AJ=J
261           BOD=BOD*(AJ+BET)
264        16 CONTINUE
266           CO=TOD/BOD
270           SUM=SUM+CO*A(N)
273        14 CONTINUE
275           A(L)=MUL*(DEL*F-SUM)
301        11 CONTINUE
304           CALL JACSER(DEL,A,BET)
306           CALL NAMPLT
307           CALL QIKSET(6.0,0.0,0.0,6.0,0.0,0.0)
313           CALL QIKSAX(3,3)
315           CALL QIKPLT(TT,BK,101)
320           CALL ENDPLT
321        10 CONTINUE
325       999 CONTINUE
325           RETURN
326           END


              SUBROUTINE JACSER(D,C,B)
  6           DIMENSION C(50),SF(50),P(50)
  6           DIMENSION BK(101),TT(101)
  6           COMMON/2/TI,TF,DT,MN,BK,TT
  6           TT(1)=0.0
  7           BK(1)=0.0
 10           LM=1
 11           T=TI
 12        12 T=T+DT
 14           X=2.*EXP(-D*T)-1.
 24           CALL JACOBI(MN,X,B,P)
```

```
      SF(1)=C(1)*P(1)
      DO 10 L=2,MN
      L1=L-1
      AL=L
      SF(L)=SF(L1)+C(L)*P(L)
   10 CONTINUE
      PRINT 97,T,X
   97 FORMAT(/////5X,*   T =*F6.3,*   X =*F10.5)
      PRINT 96
   96 FORMAT(///5X,* I    C(I)    *,5X,* N    F(T)    *)
      DO 11 I=1,6
      PRINT 95,I,C(I),I,SF(I)
   95 FORMAT(5X,I2,F10.2,5X,I2,F10.5)
   11 CONTINUE
      LM=LM+1
      BK(LM)=SF(5)
      TT(LM)=T
      IF(T.LE.TF) GO TO 12
      RETURN
      END


      SUBROUTINE JACOBI(N,X,B,PB)
C     THIS PROGRAM CALCULATES JACOBI POLYNOMIALS OF ORDER
C     K-1 WITH ARG X AND PARAMETER B GT -1
      DIMENSION PB(N)
      AN=N
      IF(AN-2.)1,2,3
    1 PB(1)=1.
      RETURN
    2 PB(1)=1.
      PB(2)=X-B*(1.-X)/2.
      RETURN
    3 BSQ=B*B
      BONE=B+1.
      PB(1)=1.
      PB(2)=X-B*(1.-X)/2.
      DO 4 K=3,N
      AK=K
      AK1=AK-1.
      AK2=AK-2.
      K1=K-1
      K2=K-2
      CO1=((2.*AK1)+B)*X
      CO1=((2.*AK2)+B)*CO1
      CO1=((2.*AK2)+BONE)*(CO1-BSQ)
      CO2=2.*AK2*(AK2+B)*((2.*AK1)+B)
      CO=2.*AK1*(AK1+B)*((2.*AK2)+B)
    4 PB(K)=(CO1*PB(K1)-CO2*PB(K2))/CO
      RETURN
      END


      SUBROUTINE SPLINE(X,Y,M,C,XINT,YINT)
      DIMENSION X(50),Y(50),C(4,50)
      IF(XINT-X(1))1,10,11
   10 YINT=Y(1)
      RETURN
   11 CONTINUE
      IF(X(M)-XINT)1,12,13
   12 YINT=Y(M)
      RETURN
   13 CONTINUE
      K=M/2
      N=M
    2 CONTINUE
      IF(X(K)-XINT)3,14,5
   14 YINT=Y(K)
      RETURN
    3 CONTINUE
      IF(XINT-X(K+1))4,15,7
   15 YINT=Y(K+1)
      RETURN
    4 CONTINUE
      YINT=(X(K+1)-XINT)*(C(1,K)*(X(K+1)-XINT)**2+C(3,K))
```

```
 54          YINT=YINT+(XINT-X(K))*(C(2,K)*(XINT-X(K))**2+C(4,K))
 65          RETURN
 65        5 CONTINUE
 65          IF(X(K-1)-XINT)6,16,17
 70        6 K=K-1
 72          GO TO 4
 72       16 YINT=Y(K-1)
 74          RETURN
 75       17 N=K
 77          K=K/2
100          GO TO 2
100        7 LL=K
102          K=(N+K)/2
103        8 CONTINUE
103          IF(X(K)-XINT)3,14,18
106       18 CONTINUE
106          IF(X(K-1)-XINT)6,16,19
111       19 N=K
113          K=(LL+K)/2
114          GO TO 8
115        1 PRINT 101
121      101 FORMAT(* OUT OF RANGE FOR INTERPOLATION   *)
121          STOP
123          END


             SUBROUTINE SPLICE(X,Y,M,C)
  7          DIMENSION X(50),Y(50),D(50),P(50),E(50),C(4,50)
  7          DIMENSION A(50,3),B(50),Z(50)
  7          MM=M-1
 11          DO 2 K=1,MM
 12          D(K)=X(K+1)-X(K)
 15          P(K)=D(K)/6.
 20        2 E(K)=(Y(K+1)-Y(K))/D(K)
 26          DO 3 K=2,MM
 27        3 B(K)=E(K)-E(K-1)
 34          A(1,2)=-1.-D(1)/D(2)
 37          A(1,3)=D(1)/D(2)
 41          A(2,3)=P(2)-P(1)*A(1,3)
 44          A(2,2)=2.*(P(1)+P(2))-P(1)*A(1,2)
 50          A(2,3)=A(2,3)/A(2,2)
 51          B(2)=B(2)/A(2,2)
 53          DO 4 K=3,MM
 54          A(K,2)=2.*(P(K-1)+P(K))-P(K-1)*A(K-1,3)
 61          B(K)=B(K)-P(K-1)*B(K-1)
 65          A(K,3)=P(K)/A(K,2)
 70        4 B(K)=B(K)/A(K,2)
 74          Q=D(M-2)/D(M-1)
 76          A(M,1)=1.+Q+A(M-2,3)
101          A(M,2)=-Q-A(M,1)*A(M-1,3)
105          B(M)=B(M-2)-A(M,1)*B(M-1)
112          Z(M)=B(M)/A(M,2)
114          MN=M-2
116          DO 6 I=1,MN
117          K=M-I
120        6 Z(K)=B(K)-A(K,3)*Z(K+1)
127          Z(1)=-A(1,2)*Z(2)-A(1,3)*Z(3)
133          DO 7 K=1,MM
135          Q=1./(6.*D(K))
140          C(1,K)=Z(K)*Q
143          C(2,K)=Z(K+1)*Q
146          C(3,K)=Y(K)/D(K)-Z(K)*P(K)
154        7 C(4,K)=Y(K+1)/D(K)-Z(K+1)*P(K)
165          RETURN
165          END
```

Torsional impact

```
      PROGRAM BETA(INPUT,OUTPUT,PUNCH,PLOT,TAPE 99=PLOT)
      REAL NON(4),F(4,4,1),G(4,4),D(4),PT(4)
      REAL B(4),C(4)
      REAL LP(50),DTA(50)
      EQUIVALENCE (NON,B)
       COMMON K1,K2,K3,K4
      COMMON/AUX/H,P,PK1,PK2,BMU,X,Y
      LP(1)=0.0
      DTA(1)=0.0
      READ 2,K1,K2,K3,K4
    2 FORMAT(I2)
*   K1 = ORDER OF SYSTEM OF EQUATIONS
*   K2 = NO. OF DISTINCT KERNELS
*   K3 = NO. OF DATA POINTS
*   K4 = NO. OF DATA SETS TO BE EVALUATED
*   SET UP DATA POINTS
      AK=K3
      DO 5 N=1,K3
      AN=N
    5 PT(N)=AN/AK
*   SET UP INTEGRATION MATRIX
      M=K3-2
      N=K3-1
      A=K3
      A=1./(3.*A)
      DO 10 K=2,M,2
   10 D(K)=2.*A
      DO 15 K=1,N,2
   15 D(K)=4.*A
      D(K3)=A
*   CALCULATE NONHOMOGENEOUS TERMS
      RHS=1.0
      DO 22 I=1,K2
      PRINT 9
    9 FORMAT(1H1)
      READ 61,BMU
   61 FORMAT(F10.5)
      DO 999 II=1,K4
      DO 35 N=1,K3
   35 NON(N)=RHS*PT(N)*PT(N)
*   CALCULATE KERNEL MATRICES
      CALL CONST(I)
      DO 20 N=1,K3
      DO 20 M=1,K3
      IF(M-N)25,30,30
   25 F(M,N,I)=F(N,M,I)
      GO TO 20
   30 F(M,N,I)=FU(I,PT(M),PT(N))
   20 CONTINUE
      CALL CHANGE(F,G,D,I)
      CALL LINEQ(G,B,C,    K3)
       DO 40 L=1,K3
      PRINT 6,PT(L),NON(L)
    6 FORMAT(5X,F8.4,F15.6)
   40 CONTINUE
      LP(II+1)=NON(K3)
      DTA(II+1)=P
  999 CONTINUE
      PUNCH 66,(DTA(IX),LP(IX),IX=1,19)
   66 FORMAT(2F10.5)
      CALL LAPINV(DTA,LP)
   22 CONTINUE
      END


      FUNCTION SIMP(I,A,B)
      COMMON/AUX/H,P,PK1,PK2,BMU,X,Y
      DEL=0.25*(B-A)
      IF(DEL)40,45,50
   45 SIMP=0.0
      RETURN
   50 CONTINUE
      SA=Z(I,A)+Z(I,B)
      SB=Z(I,A+2.*DEL)
      SC=Z(I,A+DEL)+Z(I,A+3.*DEL)
```

```
      S1=(DEL/3.)*(SA+2.*SB+4.*SC)
      IF(S1.EQ.0.0) GO TO 45
      K=8
   35 SB=SB+SC
      DEL=0.5*DEL
      SC=Z(I,A+DEL)
      J=K-1
      DO 5 N=3,J,2
      AN=N
    5 SC=SC+Z(I,A+AN*DEL)
      S2=(DEL/3.)*(SA+2.*SB+4.*SC)
      DIF=ABS((S2-S1)/S1)
      ER=0.01
      IF(DIF-ER)30,25,25
   30 SIMP=S2
      RETURN
   25 K=2*K
      S1=S2
      IF(K-2048)35,35,40
   40 PRINT 42,I,A,B
   42 FORMAT(5X,* INT. DOES NOT CONVERGE *,I3,2F9.4)
      PRINT 60,X,Y
   60 FORMAT(2F10.5)
      DO 70 J=1,10
      DIP=J
      DIP=DIP/10.
      W=Z(I,DIP)
      PRINT 60,W
   70 CONTINUE
      CALL EXIT
      END


      SUBROUTINE CHANGE(F,G,D,I)
      REAL F(4,4,1),G(4,4),D(4)
      COMMON K1,K2,K3,K4
      DO 10 N=1,K3
      DO 10 M=1,K3
      G(M,N)  =F(M,N,I)*D(N)
   10 CONTINUE
      DO 20 N=1,K3
   20 G(N,N)=G(N,N)+1.0
      RETURN
      END


      SUBROUTINE LINEQ(A,B,T,N)
      REAL A(N,N),B(N),T(N)
      DO 5 I=2,N
    5 A(I,1)=A(I,1)/A(1,1)
      DO 10 K=2,N
      M=K-1
      DO 15 I=1,N
   15 T(I)=A(I,K)
      DO 20 J=1,M
      A(J,K)=T(J)
      J1=J+1
      DO 20 I=J1,N
      T(I)=T(I)-A(I,J)*A(J,K)
   20 CONTINUE
      A(K,K)=T(K)
      IF(K.EQ.N) GO TO 10
      M=K+1
      DO 25 I=M,N
   25 A(I,K)=T(I)/A(K,K)
   10 CONTINUE
*  BACK SUBSTITUTE
      DO 30 I=1,N
      T(I)=B(I)
      M=I+1
      IF(M.GT.N) GO TO 30
      DO 30 J=M,N
      B(J)=B(J)-A(J,I)*T(I)
   30 CONTINUE
      DO 35 I=1,N
```

```
      K=N+1-I
      B(K)=T(K)/A(K,K)
      K1=K-1
      IF(K1.EQ.0) GO TO 35
      DO 35 J1=1,K1
      J=K-J1
      T(J)=T(J)-A(J,K)*B(K)
   35 CONTINUE
      RETURN
      END
```

```
      FUNCTION FU(I,A,B)
      COMMON/AUX/H,P,PK1,PK2,BMU,X,Y
      X=A
      Y=B
      IF(A*B)5,10,5
   10 FU=0.0
      RETURN
    5 SUM=SIMP(I,0.0,5.0)
      ER=0.01
      DEL =5.0
   20 UP=DEL+5.0
      ADDL=SIMP(I,DEL,UP)
      DEL =UP
      TEST=ABS(ADDL/SUM)
      SUM=SUM+ADDL
      IF(TEST-ER)15,20,20
   15 FU=SQRT(X*Y)*SUM
      RETURN
      END
```

```
      SUBROUTINE CONST(I)
      COMMON/AUX/H,P,PK1,PK2,BMU,X,Y
      PR1=0.29
      PR2=0.29
      PK1=SQRT((1.-2.*PR1)/(2.*(1.-PR1)))
      PK2=SQRT((1.-2.*PR2)/(2.*(1.-PR2)))
      READ 1,P
    1 FORMAT(F10.5)
      HH=5.0
      HH=0.2
      HH=0.5
      HH=1.0
      HH=2.0
      H=1./HH
      PRINT 2,BMU,PR1,PR2,HH,P
    2 FORMAT(/////5X,* MU2/MU1 =*F6.2,* NU1 =*F4.2,* NU2 =*F4.2///5X,* A
     1/H =*F4.2,* C21/PA =*F4.2)
      RETURN
      END
```

```
      FUNCTION Z(I,S)
      COMMON/AUX/H,P,PK1,PK2,BMU,X,Y
      BESJT(A)=SQRT(2.*A/PI)*(SIN(A)/A/A-COS(A)/A)
      PI=3.1415926
      IF(S-0.0)5,5,10
    5 Z=0.0
      RETURN
   10 CONTINUE
      PP=P*P
      GB=SQRT(S*S+1./PP)
      GD=SQRT(S*S+1./BMU/PP)
      AA=1.-BMU*GD/GB
      AB=1.+BMU*GD/GB
      AC=1.-AA/AB*EXP(-2.*GB*H)
      AD=1.+AA/AB*EXP(-2.*GB*H)
      F=GB*AC/AD
      Z=(F-S)*BESJT(S*X)*BESJT(S*Y)
      RETURN
      END
```

```
      SUBROUTINE LAPINV(GLAM,PHI)
C     THIS PROGRAM EVALUATES THE COEFFICIENTS FOR SERIES
C     OF JACOBI POLYNOMIALS WHICH REPRESENTS A LAPLACE
C     INVERSION INTEGRAL
      REAL MUL
      DIMENSION A(50),GLAM(50),PHI(50),C(4,50)
      DIMENSION BK(101),TT(101)
      COMMON/2/TI,TF,DT,MN,BK,TT
      READ 1,NN,MN,MM
    1 FORMAT(3I2)
      READ 2,TI,TF,DT
    2 FORMAT(3F10.5)
      PRINT 99
   99 FORMAT(1H1)
      CALL SPLICE(GLAM,PHI,MM,C)
      PRINT 101
  101 FORMAT(////5X,*   GLAM                PHI      *)
      PRINT 102,(GLAM(I),PHI(I),I=1,MM)
  102 FORMAT(5X,F10.5,5X,F10.5)
      M11=MM-1
      PRINT 300
  300 FORMAT(////5X,*   C(1,J)             C(2,J)             C(3,J)             C(4
     1,J) *)
      PRINT 103,((C(I,J),I=1,4),J=1,M11)
  103 FORMAT(5X,F10.5,5X,F10.5,5X,F10.5,5X,F10.5)
      PRINT 99
      DO 10 I=1,NN
      READ 3,BET,DEL
    3 FORMAT(2F10.5)
      PRINT 98,BET,DEL
   98 FORMAT(/////5X,*BETA =*F5.3,* DELTA =*F5.3)
      DO 11 L=1,MN
      AL=L
      S=1./(AL+BET)/DEL
      CALL SPLINE(GLAM,PHI,MM,C,S,G)
      F=G*S
      IF(AL-2.)81,82,83
   81 A(1)=(1.+BET)*DEL*F
      GO TO 11
   82 A(2)=((2.+BET)*DEL*F-A(1))*(3.+BET)
      GO TO 11
   83 CONTINUE
      TOP=1.
      L1=L-1
      AL1=L1
      DO 12 J=1,L1
      AJ=J
      TOP=AJ*TOP
   12 CONTINUE
      L2=2*L-1
      BOT=1.
      DO 13 J=L,L2
      AJ=J
      BOT=(AJ+BET)*BOT
   13 CONTINUE
      MUL=BOT/TOP
      SUM=0.0
      DO 14 N=1,L1
      AN=N
      IF(AN-2.)85,86,87
   85 TOD=1.
      GO TO 88
   86 TOD=AL1
      GO TO 88
   87 CONTINUE
      TOD=1.
      ICH=L1-(N-2)
      DO 15 J=ICH,L1
      AJ=J
      TOD=AJ*TOD
   15 CONTINUE
   88 CONTINUE
      BOD=1.
      JA=L1+N
      DO 16 J=L,JA
      AJ=J
```

```
      BOD=BOD*(AJ+BET)
   16 CONTINUE
      CO=TOD/BOD
      SUM=SUM+CO*A(N)
   14 CONTINUE
      A(L)=MUL*(DEL*F-SUM)
   11 CONTINUE
      CALL JACSER(DEL,A,BET)
      CALL NAMPLT
      CALL QIKSET(6.0,0.0,0.0,6.0,0.0,0.0)
      CALL QIKSAX(3,3)
      CALL QIKPLT(TT,BK,101)
      CALL ENDPLT
   10 CONTINUE
  999 CONTINUE
      RETURN
      END


      SUBROUTINE JACSER(D,C,B)
      DIMENSION C(50),SF(50),P(50)
      DIMENSION BK(101),TT(101)
      COMMON/2/TI,TF,DT,MN,BK,TT
      TT(1)=0.0
      BK(1)=0.0
      LM=1
      T=TI
   12 T=T+DT
      X=2.*EXP(-D*T)-1.
      CALL JACOBI(MN,X,B,P)
      SF(1)=C(1)*P(1)
      DO 10 L=2,MN
      L1=L-1
      AL=L
      SF(L)=SF(L1)+C(L)*P(L)
   10 CONTINUE
      PRINT 97,T,X
   97 FORMAT(/////5X,*   T =*F6.3,*  X =*F10.5)
      PRINT 96
   96 FORMAT(///5X,* I    C(I)     *,5X,* N    F(T)     *)
      DO 11 I=1,6
      PRINT 95,I,C(I),I,SF(I)
   95 FORMAT(5X,I2,F10.2,5X,I2,F10.5)
   11 CONTINUE
      LM=LM+1
      BK(LM)=SF(5)
      TT(LM)=T
      IF(T.LE.TF) GO TO 12
      RETURN
      END


      SUBROUTINE JACOBI(N,X,B,PB)
C     THIS PROGRAM CALCULATES JACOBI POLYNOMIALS OF ORDER
C     K-1 WITH ARG X AND PARAMETER B GT -1
      DIMENSION PB(N)
      AN=N
      IF(AN-2.)1,2,3
    1 PB(1)=1.
      RETURN
    2 PB(1)=1.
      PB(2)=X-B*(1.-X)/2.
      RETURN
    3 BSQ=B*B
      BONE=B+1.
      PB(1)=1.
      PB(2)=X-B*(1.-X)/2.
      DO 4 K=3,N
      AK=K
      AK1=AK-1.
      AK2=AK-2.
      K1=K-1
      K2=K-2
      CO1=((2.*AK1)+B)*X
      CO1=((2.*AK2)+B)*CO1
```

```
      CO1=((2.*AK2)+BONE)*(CO1-BSQ)
      CO2=2.*AK2*(AK2+B)*((2.*AK1)+B)
      CO=2.*AK1*(AK1+B) *((2.*AK2)+B)
    4 PB(K)=(CO1*PB(K1) -CO2*PB(K2))/CO
      RETURN
      END


      SUBROUTINE SPLINE(X,Y,M,C,XINT,YINT)
      DIMENSION X(50),Y(50),C(4,50)
      IF(XINT-X(1))1,10,11
   10 YINT=Y(1)
      RETURN
   11 CONTINUE
      IF(X(M)-XINT)1,12,13
   12 YINT=Y(M)
      RETURN
   13 CONTINUE
      K=M/2
      N=M
    2 CONTINUE
      IF(X(K)-XINT)3,14,5
   14 YINT=Y(K)
      RETURN
    3 CONTINUE
      IF(XINT-X(K+1))4,15,7
   15 YINT=Y(K+1)
      RETURN
    4 CONTINUE
      YINT=(X(K+1)-XINT)*(C(1,K)*(X(K+1)-XINT)**2+C(3,K))
      YINT=YINT+(XINT-X(K))*(C(2,K)*(XINT-X(K))**2+C(4,K))
      RETURN
    5 CONTINUE
      IF(X(K-1)-XINT)6,16,17
    6 K=K-1
      GO TO 4
   16 YINT=Y(K-1)
      RETURN
   17 N=K
      K=K/2
      GO TO 2
    7 LL=K
      K=(N+K)/2
    8 CONTINUE
      IF(X(K)-XINT)3,14,18
   18 CONTINUE
      IF(X(K-1)-XINT)6,16,19
   19 N=K
      K=(LL+K)/2
      GO TO 8
    1 PRINT 101
  101 FORMAT(* OUT OF RANGE FOR INTERPOLATION  *)
      STOP
      END


      SUBROUTINE SPLICE(X,Y,M,C)
      DIMENSION X(50),Y(50),D(50),P(50),E(50),C(4,50)
      DIMENSION A(50,3),B(50),Z(50)
      MM=M-1
      DO 2 K=1,MM
      D(K)=X(K+1)-X(K)
      P(K)=D(K)/6.
    2 E(K)=(Y(K+1)-Y(K))/D(K)
      DO 3 K=2,MM
    3 B(K)=E(K)-E(K-1)
      A(1,2)=-1.-D(1)/D(2)
      A(1,3)=D(1)/D(2)
      A(2,3)=P(2)-P(1)*A(1,3)
      A(2,2)=2.*(P(1)+P(2))-P(1)*A(1,2)
      A(2,3)=A(2,3)/A(2,2)
      B(2)=B(2)/A(2,2)
      DO 4 K=3,MM
      A(K,2)=2.*(P(K-1)+P(K))-P(K-1)*A(K-1,3)
      B(K)=B(K)-P(K-1)*B(K-1)
```

```
 65          A(K,3)=P(K)/A(K,2)
 70       4  B(K)=B(K)/A(K,2)
 74          Q=D(M-2)/D(M-1)
 76          A(M,1)=1.+Q+A(M-2,3)
101          A(M,2)=-Q-A(M,1)*A(M-1,3)
105          B(M)=B(M-2)-A(M,1)*B(M-1)
112          Z(M)=B(M)/A(M,2)
114          MN=M-2
116          DO 6 I=1,MN
117          K=M-I
120       6  Z(K)=B(K)-A(K,3)*Z(K+1)
127          Z(1)=-A(1,2)*Z(2)-A(1,3)*Z(3)
133          DO 7 K=1,MM
135          Q=1./(6.*D(K))
140          C(1,K)=Z(K)*Q
143          C(2,K)=Z(K+1)*Q
146          C(3,K)=Y(K)/D(K)-Z(K)*P(K)
154       7  C(4,K)=Y(K+1)/D(K)-Z(K+1)*P(K)
165          RETURN
165          END
```

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